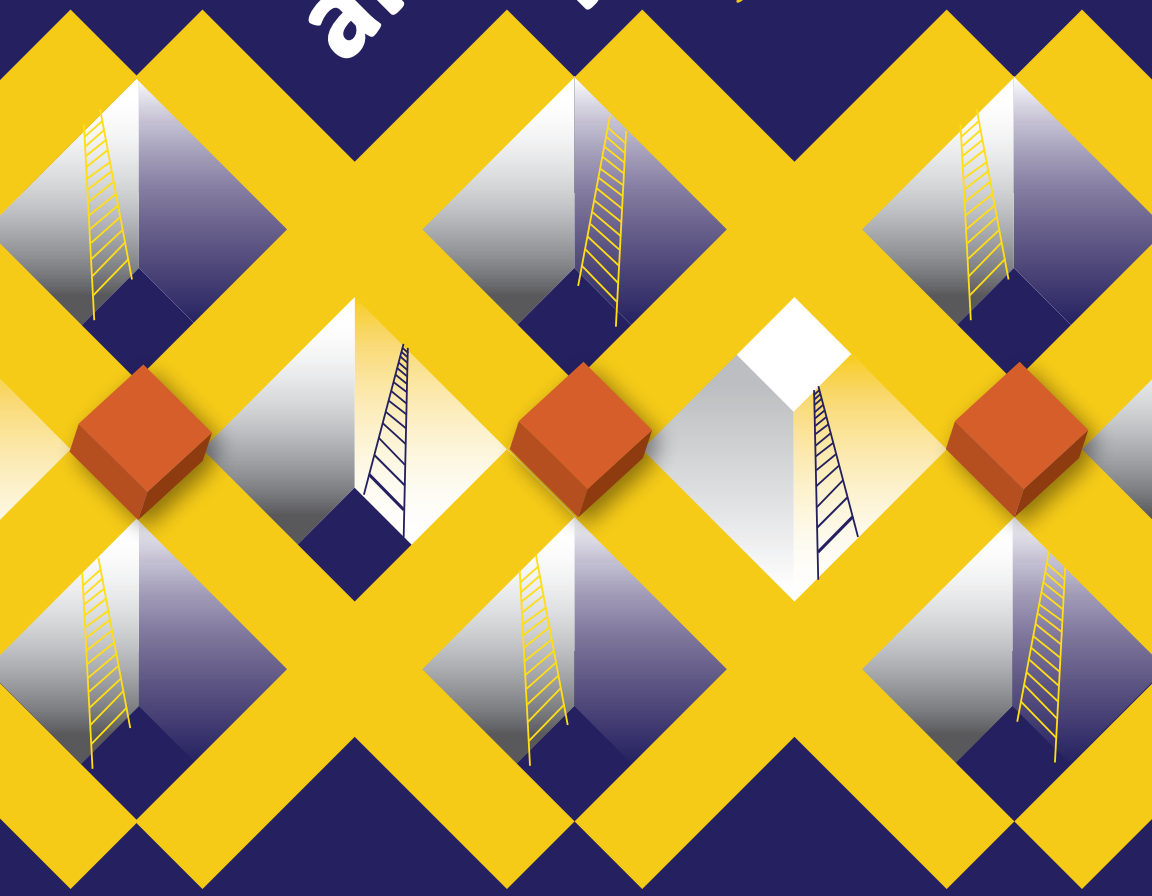
Paradoxes and Sophisms in Calculus

Sergiy Klymchuk
Susan Staples



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Classroom Resource Materials

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To our parents and families

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Introduction

Dear God,
If I have just one hour remaining to live,
Please put me in a calculus class
So that it will seem to last forever.
— A bored student's prayer

In the study of mathematics non-routine problems, puzzles, paradoxes, and sophisms often delight and fascinate. Captivating examples can excite, enlighten, and inspire learners and spur their passion for discovery. Furthermore, thought-provoking exercises and contemplation of paradoxes can naturally engage students and offer them a unique opportunity to understand more fully the history and development of mathematics. "Justification of otherwise inexplicable notions on the grounds that they yield useful results has occurred frequently in the evolution of mathematics [15]."

The teaching and learning process often loses its effectiveness for lack of appropriate intellectual challenges and for insufficient active involvement or emotional investment from students in the experience. What we tend to remember most are knowledge and learning experiences tied to intense thinking, noteworthy discovery, or inspired creativity. This book presents problems and examples that may lead students to contemplate conceptual issues in calculus and to comprehend the subtleties of this subject more deeply.

In that light, this book aims to enhance the teaching and learning of a first-year calculus course. The following major topics from a typical single-variable calculus course are explored in the book: functions, limits, derivatives, and integrals.

The book consists of two main types of examples: paradoxes and sophisms. Why should we endeavor to study these atypical or troublesome problems from calculus? Consider these compelling remarks from *How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics* [6, p. 6]: "Logic abhors the ambiguous, the paradoxical, and especially the contradictory, but the creative mathematician welcomes such problematic situations because they raise the question, 'What is going on here?' Thus the problematic signals a situation that is worth investigating. The problematic is a potential source of new mathematics."

The word *paradox* comes from the Greek word *paradoxon* which means *unexpected*. Several usages of this word exist, including those that allow for contradiction. However, in this book, the word paradox will exclusively be used to mean a surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true. "All I know is that I know nothing," a statement attributed to Socrates in Plato's *Republic*, offers a classical paradox of logic. In this book the paradoxes we examine relate to notions in calculus or the study of functions and limits. While some of the paradoxes presented in the book (such as "A cat on a ladder" and "Encircling the Earth") may more naturally be considered precalculus topics, they can still be discussed in calculus classes to demonstrate that intuition can fail, even when we are considering examples with shapes as mundane as circles or spheres. On the other hand, while all the paradoxes or problems we examine can be understood in a first-year calculus class, the explanations for a few problems may involve topics more traditionally classified as advanced calculus or elementary analysis.

A number of the paradoxes touch upon classical examples, specialized functions, or canonical curves. In such instances, we mention the paradoxes or specific examples by name, in either the initial presentation or the solution of the paradox, whichever place seems more fitting. If the nature of the paradox depends on the definition of the precise curve, such as in the paradoxes involving the Koch snowflake or the Sierpinski carpet, we will include the curve name or paradox name at the outset. On the other hand, at times where we prefer to keep the option of open-ended discovery available, we may not provide the classical terminology until the solution is presented. When possible, we have also included references that treat related material in an expository or introductory manner. In this way, we hope to offer readers resources for projects, classroom presentations, or subsequent inquiry. Much of the book's content can be viewed as recreational mathematics and can be used as a natural stepping stone to further investigations into mathematical topics.

The word *sophism* comes from the Greek word *Sophos* which means *wisdom*. In modern usage it denotes intentionally invalid reasoning that looks formally correct, but in fact contains a subtle mistake or flaw. In other words, a sophism is a false proof of an incorrect statement. Each such "proof" contains some sort of error in reasoning. Plato in his desire to pursue the truth found nothing more deplorable than Sophists using deliberately deceptive arguments for personal empowerment. Priestley [29, pp. 75–80] provides a pleasant overview of this history in *Calculus: A Liberal Art*.

Many students are exposed to sophisms at school. The exercise of finding and analyzing the mistake in a sophism often provides a deeper understanding than a mere recipe-based approach in solving problems. Typical algebraic examples of flawed reasoning that produce sophisms include division by zero or taking only a nonnegative square root. The following sophism from basic algebra utilizes the trick of division by zero to "prove" that "$1 = 2$".

Statement: If $x = y$, then $1 = 2$.

Proof:

$$x = y$$

Thus

$$xy = y^2,$$

And

$$xy - x^2 = y^2 - x^2,$$

so that

$$x(y - x) = (y + x)(y - x).$$

Now dividing by $y - x$ gives

$$x = y + x.$$

Then substituting $x = y$ on the right gives

$$x = 2x.$$

Finally dividing by x gives

$$1 = 2.$$

In this book the tricks or incorrect reasoning steps that lead to the sophisms are tied to calculus concepts. The examples are designed to reinforce the correct understanding of oft-misconstrued principles. According to our usage of the terms paradox and sophism, many well-known so-called paradoxes, such as Zeno's paradoxes and Aristotle's wheel paradox will be viewed as sophisms in this book. When classical sophisms are treated, we

provide references when possible. We hope that interested readers will use the ideas in this book as a springboard for future mathematical investigations.

In a manner similar to the first author's previous book, *Counterexamples in Calculus* [17] this book aspires to encourage students and teachers to examine paradoxes and sophisms that arise in calculus for these purposes:

- To provide deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance mathematical thinking beyond algorithmic or procedural reasoning
- To enhance baseline critical thinking skills—analyzing, justifying, verifying, and checking
- To expand the example set of noteworthy mathematical ideas
- To engage students in more active and creative learning
- To encourage further investigation of mathematical topics

In that regard, this book may well serve

- High school teachers and university faculty as a teaching resource
- High school and college students as a learning resource for calculus
- Calculus instructors as a professional development resource

Contact Details for Feedback

Please send your questions and comments regarding the book to the authors.

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No success is truly one's alone; triumph is always brought about and shared by family.

Part I

Paradoxes

I see it but I don't believe it!

— Georg Cantor (1845–1918),
in a letter to Richard Dedekind (1877)

1

Functions and Limits

1 Laying bricks

Imagine you have an unlimited supply of identical ideal homogeneous bricks. You construct an arch by putting the bricks one on top of another without applying any cement between layers. Each successive brick is placed further to the right than the previous (see Figure 1.1).

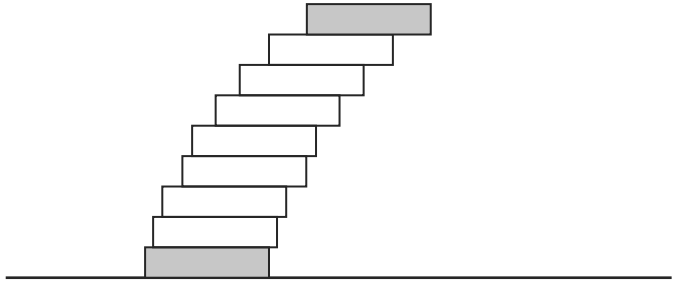


Figure 1.1

How far past the bottom brick can the top brick extend?

2 Spiral curves

Construct two similar-looking spiral curves (see Figure 1.2) that both rotate infinitely many times around a point, with one curve being of a finite length

and the other of an infinite length.

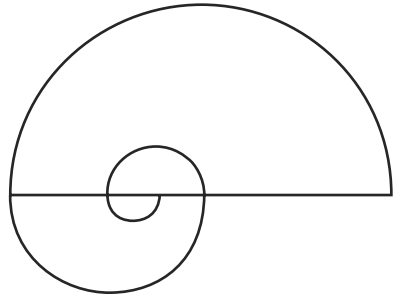


Figure 1.2

3 A paradoxical fractal curve: the Koch snowflake

An equilateral triangle of side length 1 unit is transformed recursively as follows. First mark the middle third of each side of the previous stage. Next, construct outward facing equilateral triangles with each of these middle portions as bases. Finally remove these segments that served as bases. Iterate this process indefinitely. The first few stages are depicted in Figure 1.3. The Koch snowflake is the curve defined as the limit of this process. Show that this curve has infinite length, yet it is located between two other closed, non self-intersecting curves of finite length.

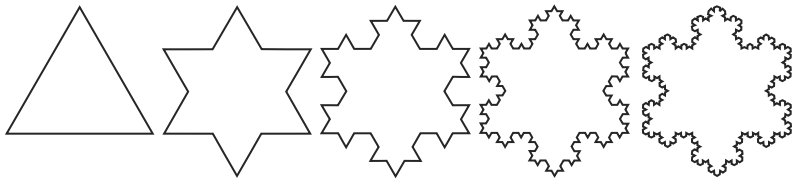


Figure 1.3

4 A tricky fractal area: the Sierpinski carpet

A square with sides of 1 unit (and therefore area of 1 square unit) is divided into nine equal squares, each with sides $1/3$ unit and areas of $1/9$ square unit, then the central square is removed. Denote the original square by $C(0)$ and the subsequent stage as $C(1)$. Each of the remaining eight squares is

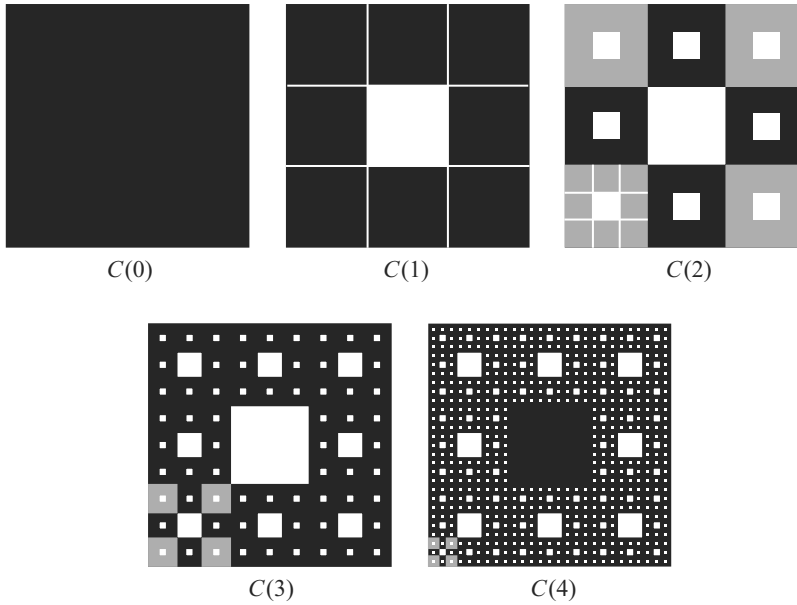


Figure 1.4

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divided into nine equal squares and the central squares are then removed. Label the newly produced stage as $C(2)$. Iterate the process indefinitely. Figure 1.4 shows the first four steps. At every step $1/9$ of the current area is removed and $8/9$ is left, that is at every step the remaining area is eight times bigger than the area removed. After infinitely many steps what would the remaining area be?

5 A mysterious fractal set: the Cantor ternary set

Start with the unit interval, $C(0) = [0, 1]$ and divide this into three equal subintervals. Remove the open middle third interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Call what remains the set $C(1) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. At the next step of the construction remove the middle open third of each of these two intervals. This results in $C(2) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Iterate this process indefinitely. Figure 1.5 illustrates the first four stages.

The Cantor ternary set C consists of all the points in the original interval $[0, 1]$ that are not removed at any stage. That is, $C = \lim_{n\to\infty} C(n)$. A striking pair of paradoxical facts holds true for the Cantor ternary set. Compute the proportion of the original interval $[0, 1]$ contained in the nth



Figure 1.5

stage Cantor set $C(n)$ and show this proportion goes to 0 as $n \to \infty$. On the other hand, prove that C contains as many points as the original interval $[0, 1]$! (Hint: As suggested by the name of the set, you may want to think about ternary expansions of numbers.)

6 A misleading sequence

What is the next term in the sequence 2, 4, 8, 16?

7 Remarkable symmetry

For a shape with a center of symmetry, define a diameter as a line segment that passes through the center and joins two opposite boundary points. For example, Figure 1.6 pictures two diameters d_1 and d_2 for a regular hexagon and a circle.

Given a shape with a center of symmetry, a student hypothesized that if all diameters of the shape were equal in length, then the shape must be a circle. Was the student's reasoning correct?

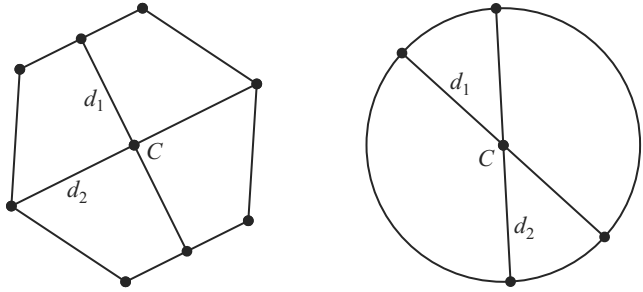


Figure 1.6

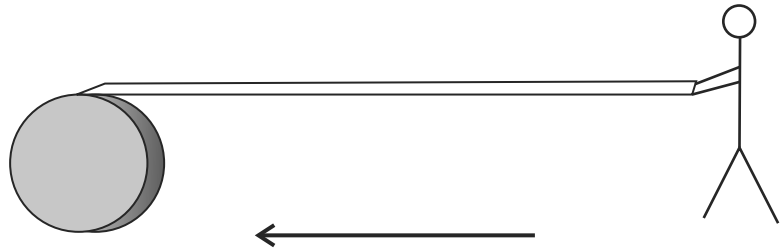


Figure 1.7

8 Rolling a barrel

A person holds one end of a wooden board 3 m long and the other end lies on a cylindrical barrel. The person walks towards the barrel, which is rolled by the board sitting on it. The barrel rolls without sliding and the board remains parallel to the ground. This process is illustrated in Figure 1.7.

What distance will the person cover before reaching the barrel?

9 A cat on a ladder

Imagine a cat sitting half way up a ladder that is placed almost flush with a wall. (See Figure 1.8.)

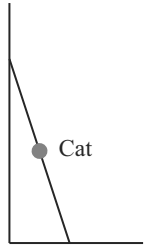


Figure 1.8

Part 1 If the base of the ladder is pushed fully up against the wall, the ladder and cat are most likely going to fall away from the wall (i.e., the top of the ladder falls away from the wall).

If the cat stays on the ladder (not likely perhaps?) what will the trajectory of the cat be? A, B or C in Figure 1.9?

Part 2 Which of the Figure 1.9 options represents the cat's trajectory if instead of the top of the ladder falling outwards, the base is pulled away? A, B or C?

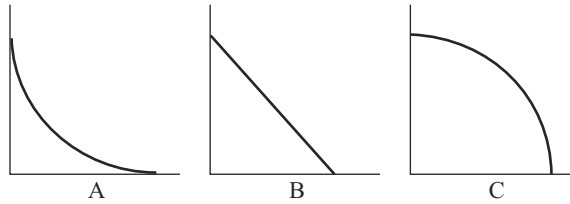


Figure 1.9

10 Sailing

A yacht returns from a trip around the world. Different parts of the yacht have covered different distances. Which part of the yacht has covered the longest distance?

11 Encircling the Earth

Imagine a rope lying around the Earth's equator without any bends (i.e., idealize the Earth as a sphere and ignore mountains and deep-sea trenches). The rope is lengthened by 20 meters and the circle is formed again. Estimate approximately how high the rope will be above the Earth.

(a) 3 mm (b) 3 cm (c) 3 m

12 A tricky equation

To check the number of solutions of the equation $\log_{1/16} x = \left(\frac{1}{16}\right)^x$ one can sketch the graphs of this pair of inverse functions $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$. See Figure 1.10.

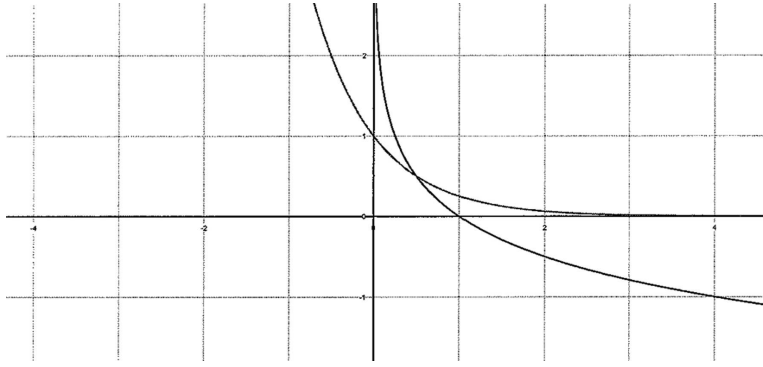


Figure 1.10

From the graphs we can see that there is one intersection point and therefore one solution to the equation $\log_{1/16} x = \left(\frac{1}{16}\right)^x$. But it is easy to check by substitution that both $x = \frac{1}{2}$ and $x = \frac{1}{4}$ satisfy the equation. So how many solutions does the equation have?

13 A snail on a rubber rope

Imagine a snail moving at a speed of 1 cm/min along a rubber rope 1 m long. The snail starts its journey from one end of the rope. After each minute the rope is uniformly expanded by 1 m. With each stretching, the snail is carried forward with the elongated rope. Will the snail ever reach the end of the rope?

2

Derivatives and Integrals

1 An alternative product rule

A novice calculus student believes the derivative of the product of two differentiable functions to be the product of their derivatives: $(uv)' = u'v'$. Clearly, if either of u or v is the zero function, or if both u and v are constant functions, then the rule holds. Show that, strangely enough, there are infinitely many other pairs of functions u and v for which this product rule holds true.

2 Missing information?

At first glance it appears there is not enough information to solve the following problem: Given a metal sphere of radius greater than 8 cm, drill a circular hole of 16 cm through its center. Find the volume of the remaining part of the sphere.

3 A paint shortage

To paint the area bounded by the curve $y = \frac{1}{x}$, the x-axis, and the line $x = 1$ is impossible. There is not enough paint in the world, because the area is infinite:

$$\int_1^\infty \frac{1}{x} dx = \lim_{b \to \infty} (\ln b - \ln 1) = \infty.$$

However, one can rotate the area around the x-axis and the resulting solid of revolution would have a finite volume:

$$\pi \int_1^{\infty} \frac{1}{x^2} dx = -\pi \lim_{b \to \infty} \left(\frac{1}{b} - \frac{1}{1} \right) = \pi.$$

This solid of revolution contains the area, which is a cross-section of the solid. One can fill the solid with π cubic units of paint and thus cover the area with paint. Can you explain this paradox?

4 Racing marbles

Consider the curve given by the parametric equations

$$x(\theta) = \theta - \sin\theta \quad \text{and} \quad y(\theta) = \cos\theta - 1,$$

for $0 \leq \theta \leq \pi$. The curve starts at the origin and ends at the point $(\pi, -2)$ as shown in Figure 2.1.

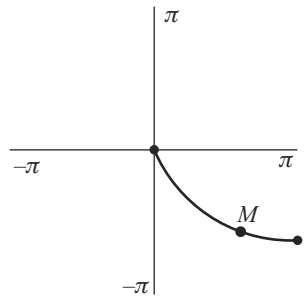


Figure 2.1

Suppose that a marble is set at any point M on this curve. Amazingly the amount of time the marble needs to roll to the finish point at $(\pi, -2)$ is independent of M! How can this be true?

5 A paradoxical pair of functions

The graphs of the two functions

$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

appear in Figures 2.2 and 2.3.

The graphs appear to exhibit similar oscillatory behavior near $x = 0$. In fact, it is possible to show that both $f(x)$ and $g(x)$ are continuous at $x = 0$.

However, $f(x)$ is not differentiable at $x = 0$, while $g(x)$ is differentiable at $x = 0$. How can this be true?

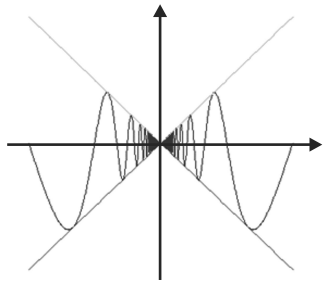


Figure 2.2 $y = x \sin\left(\frac{1}{x}\right)$

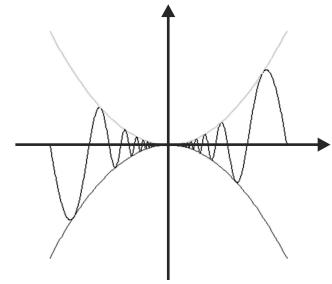


Figure 2.3 $y = x^2 \sin\left(\frac{1}{x}\right)$

6 An unruly function

A well-known calculus theorem states that if a function $f(x)$ is differentiable on an interval (a, b), then $f(x)$ is continuous on (a, b) as well. Is it possible to find a function that is not continuous on any interval and yet it is differentiable at some point?

7 Jagged peaks galore

The standard example $f(x) = |x|$ offers a continuous function that is not differentiable for a certain value of x. Here $f(x)$ fails to be differentiable at $x = 0$, where the graph of the function contains a corner. It is easy to imagine that adding more peaks or corners to the graph of $f(x)$ could induce non-differentiability at more values of x. What is the most extreme manner in which a continuous function can fail to be differentiable?

This question persisted in the minds of mathematicians for much of the development of calculus:

"There was a presumption in the seventeenth through mid-nineteenth centuries that all continuous functions can be differentiated, with perhaps a few exceptional points (such as interface points for functions defined in pieces). In fact, Joseph Louis Lagrange (1736–1813) built a whole theory of functions around this assumption." [21, p. 136]

However, this presumption in the history of calculus fails to be true in a most striking way. Spectacular pathological functions exist that are continuous everywhere, but differentiable nowhere! Can you envision this implausible phenomenon?

8 Another paradoxical pair of functions

Consider two functions defined on $[0, 1]$. The first of these is Dirichlet's function,

$$f(x) = \begin{cases} 1 & \text{for } x \text{ a rational number} \\ 0 & \text{for } x \text{ an irrational number} \end{cases}.$$

For the second function $g(x)$ we again split our definition into cases based on whether the value x is rational or irrational. When x is rational, we first write it in reduced form, that is $x = p/q$, where the integers p and q share no common factors. Here we consider the value 0 to be in lowest terms when written as $0 = 0/1$. We define the function as

$$g(x) = \begin{cases} 1/q & \text{for } x \text{ rational, } x = p/q \\ 0 & \text{for } x \text{ an irrational number} \end{cases}.$$

For instance, $g(0) = g(1) = 1$, and $g(\frac{9}{12}) = g(\frac{3}{4}) = \frac{1}{4}$. Beyond the similar mode of definition, another feature shared by $f(x)$ and $g(x)$ is their discontinuity at each rational point x. However, $f(x)$ is not Riemann integrable, while $g(x)$ is! How can this difference be explained?

Part II

Sophisms

1 + 1 = 3 for large values of 1.

— A student joke

3

Functions and Limits

1 Evaluation of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$ proves that $1 = 0$.

We determine the limit using two different methods and equate the values.

(a) Applying the theorem for the limit of a sum, we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \\
&\quad = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2}} + \cdots + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} \\
&\quad = 0 + 0 + \cdots + 0 = 0.
\end{aligned}$$

(b) For our second approach to the problem, we first find lower and upper bounds for the sum

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}.$$

We have

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}}.$$

Both the lower and upper bounds converge to 1 as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1.$$

It follows by the squeeze theorem that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = 1.$$

Comparing the results from (a) and (b), we conclude that $1 = 0$.

2 Evaluation of $\lim_{x\to 0}\left(x \sin \frac{1}{x}\right)$ proves that $1 = 0$.

We find $\lim_{x\to 0}\left(x \sin \frac{1}{x}\right)$ using two methods and compare our answers.

(a) Since $-1 \le \sin \frac{1}{x} \le 1$ we have $-|x| \le x \sin \frac{1}{x} \le |x|$. Applying the squeeze theorem we conclude that $\lim_{x\to 0}\left(x \sin \frac{1}{x}\right) = 0$.

(b) It is well known that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Rewriting the limit we obtain

$$\lim_{x\to 0}\left(x \sin \frac{1}{x}\right) = \lim_{x\to 0} \frac{\sin(1/x)}{(1/x)} = 1.$$

Equating the results found in (a) and (b), we conclude that $1 = 0$.

3 Evaluation of $\lim_{x\to 0^+}(x^x)$ shows that $1 = 0$.

Again we determine the limit using two methods.

(a) First find the limit of the base and then evaluate the remaining portion of the limit. We then have

$$\lim_{x\to 0^+} (x^x) = \left(\lim_{x\to 0^+} x\right)^x = \lim_{x\to 0^+} 0^x = 0.$$

(b) For our second solution, we first find the limit of the power and then determine the remaining portion of the limit.

$$\lim_{x\to 0^+}(x^x) = x^{(\lim_{x\to 0^+} x)} = \lim_{x\to 0^+} x^0 = 1.$$

Comparing the results found in (a) and (b), we conclude that $1 = 0$.

4 Evaluation of $\lim_{n\to\infty} \sqrt[n]{n}$ demonstrates that $1 = \infty$.

As before, we compute the limit in two different manners.

(a) First find the limit of the expression under the radical sign and then evaluate the other part of the limit. By this method we have

$$\lim_{n\to\infty} \sqrt[n]{n} = \sqrt[n]{\lim_{n\to\infty} n} = \lim_{n\to\infty} \sqrt[n]{\infty} = \infty.$$

(b) This time first find the limit of the nth root and then determine the other part of the limit. This solution gives

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = n^{\lim_{n\to\infty} 1/n} = \lim_{n\to\infty} n^0 = \lim_{n\to\infty} 1 = 1.$$

Comparing the results, we conclude that $1 = \infty$.

5 Trigonometric limits prove that $\sin kx = k \sin x$.

It is well known that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Using this we find both

$$\lim_{x\to 0} \frac{\sin kx}{x} = k \lim_{x\to 0} \frac{\sin kx}{kx} = k \lim_{u\to 0} \frac{\sin u}{u} = k,$$

and

$$\lim_{x\to 0} \frac{k \sin x}{x} = k \lim_{x\to 0} \frac{\sin x}{x} = k.$$

Therefore we have

$$\lim_{x\to 0} \frac{\sin kx}{x} = \lim_{x\to 0} \frac{k \sin x}{x}.$$

It follows that

$$\frac{\sin kx}{x} = \frac{k \sin x}{x},$$

which implies $\sin kx = k \sin x$.

6 Evaluation of a limit of a sum proves that $1 = 0$.

(a) We know that the limit of the sum of two sequences equals the sum of their limits, provided both limits exist. That is, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, provided both limits on the right side exist. We also know that this is true for any number k of sequences in the sum. Let us take n constant sequences $\frac{1}{n}$ and find the limit of their sum when $n \to \infty$:

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n} + \cdots + \lim_{n\to\infty}\frac{1}{n}$$
$$= 0 + 0 + \cdots + 0 = 0.$$

(b) On the other hand, the sum $\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right)_{(n \text{ terms})}$ is equal to $n \times \frac{1}{n} = 1$.

So we have shown $1 = 0$.

7 Analysis of the function $\frac{x+y}{x-y}$ proves that $1 = -1$.

Let us find two limits for this function:

(a) $$\lim_{x\to\infty}\lim_{y\to\infty}\frac{x+y}{x-y} = \lim_{x\to\infty}\lim_{y\to\infty}\frac{\frac{x}{y}+1}{\frac{x}{y}-1} = \lim_{x\to\infty}(-1) = -1.$$

(b) $$\lim_{y\to\infty}\lim_{x\to\infty}\frac{x+y}{x-y} = \lim_{y\to\infty}\lim_{x\to\infty}\frac{1+\frac{y}{x}}{1-\frac{y}{x}} = \lim_{y\to\infty}1 = 1.$$

Because

$$\lim_{x\to\infty}\lim_{y\to\infty}\frac{x+y}{x-y} = \lim_{y\to\infty}\lim_{x\to\infty}\frac{x+y}{x-y},$$

the results from (a) and (b) must be equal. Therefore we have proved that $1 = -1$.

8 Analysis of the function $\frac{ax+y}{x+ay}$ proves that $a = \frac{1}{a}$, for any value $a \neq 0$.

Let a be a nonzero number. We compute two equivalent limits

$$\lim_{x\to\infty}\lim_{y\to\infty}\frac{ax+y}{x+ay} = \lim_{y\to\infty}\lim_{x\to\infty}\frac{ax+y}{x+ay}.$$

(a) First compute

$$\lim_{x\to\infty}\lim_{y\to\infty}\frac{ax+y}{x+ay}=\lim_{x\to\infty}\lim_{y\to\infty}\frac{\frac{ax}{y}+1}{\frac{x}{y}+a}=\lim_{x\to\infty}\frac{1}{a}=\frac{1}{a}.$$

(b) Next we find

$$\lim_{y\to\infty}\lim_{x\to\infty}\frac{ax+y}{x+ay}=\lim_{y\to\infty}\lim_{x\to\infty}\frac{a+\frac{y}{x}}{1+\frac{ay}{x}}=\lim_{y\to\infty}a=a.$$

Equating the results from (a) and (b) we conclude that $a = 1/a$.

9 One-to-one correspondences imply that $1 = 2$.

(a) Consider the line segment joining $(0, 0)$ to $(2, 0)$ and the one joining $(0, 1)$ to $(1, 1)$. They have lengths 2 and 1 units respectively. We establish a one-to-one correspondence between their points as shown in Figure 3.1.

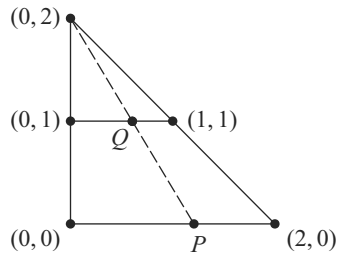


Figure 3.1

In particular, observe that $P = (x, 0)$ in the longer segment corresponds to the point $Q = (x/2, 1)$ in the shorter segment for $0 \le x \le 2$.

It follows that the number of points on the line segment of length 1 unit is the same as the number of points on the line segment of length 2 units, meaning that $1 = 2$.

(b) Take two circles of radius 1 unit and 2 units. As in part (a) define a one-to-one correspondence between the points on the circles as shown in Figure 3.2.

Because the number of points on the circumference of the inner circle is the same as the number of points on the circumference of the outer circle, we can conclude that $1 = 2$.

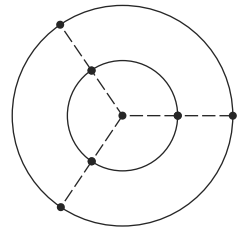


Figure 3.2

10 Aristotle's wheel implies that $R = r$.

Two wheels of different radii are attached to each other and put on the same axis. Both wheels are on a rail (see Figures 3.3 and 3.4). After one rotation the large wheel with radius R covers the distance AB which is equal to the length of its circumference $2\pi R$. The small wheel with radius r covers the distance CD which is equal to the length of its circumference $2\pi r$. Since $AB = CD$, therefore $2\pi R = 2\pi r$ and thus $R = r$.

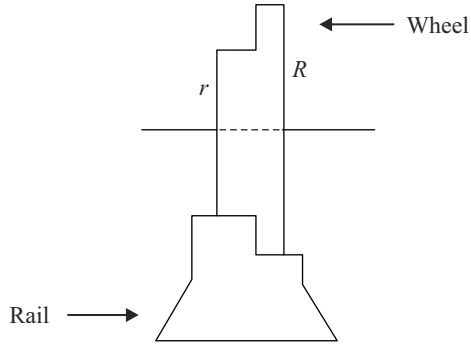


Figure 3.3 Cross-section showing the wheel and the rail.

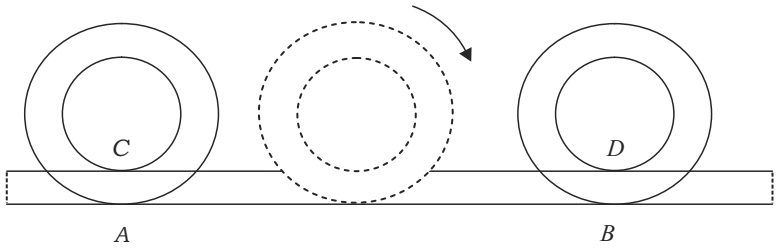


Figure 3.4

11 Logarithmic inequalities show $2 > 3$.

From

$$\frac{1}{4} > \frac{1}{8} \quad \text{or} \quad \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3,$$

taking natural logarithms of both sides yields

$$\ln\left(\frac{1}{2}\right)^2 > \ln\left(\frac{1}{2}\right)^3.$$

Applying the power rule of logarithms we have

$$2\ln\left(\frac{1}{2}\right) > 3\ln\left(\frac{1}{2}\right).$$

Dividing both sides by $\ln\left(\frac{1}{2}\right)$ then gives $2 > 3$.

12 Analysis of the logarithm function implies $2 > 3$.

From

$$\frac{1}{4} > \frac{1}{8} \quad \text{or} \quad \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3,$$

taking logarithms with base $\frac{1}{2}$ of both sides yields

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^2 > \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^3.$$

Applying the power rule of logarithms we have

$$2\log_{\frac{1}{2}}\left(\frac{1}{2}\right) > 3\log_{\frac{1}{2}}\left(\frac{1}{2}\right).$$

Finally, since $\log_{1/2}\left(\frac{1}{2}\right) = 1$ we obtain $2 > 3$.

13 Analysis of the logarithm function proves $\frac{1}{4} > \frac{1}{2}$.

From

$$\ln\frac{1}{2} = \ln\frac{1}{2},$$

doubling the left-hand side yields the inequality

$$2\ln\frac{1}{2} > \ln\frac{1}{2}.$$

Applying the power rule of logarithms we have

$$\ln\left(\frac{1}{2}\right)^2 > \ln\left(\frac{1}{2}\right).$$

Since $y = \ln x$ is an increasing function, it follows that

$$\left(\frac{1}{2}\right)^2 > \frac{1}{2} \quad \text{or} \quad \frac{1}{4} > \frac{1}{2}.$$

14 Limit of perimeter curves shows that $2 = 1$.

We start with an equilateral triangle with sides of 1 unit. Divide each of the upper sides in half and transform them into a zig-zag curve with four linear subportions as shown in Figure 3.5(b). To obtain Figure 3.5(c), we halve all the non-horizontal segments in (b) and replace them with a zig-zag piece. Continue this process indefinitely.

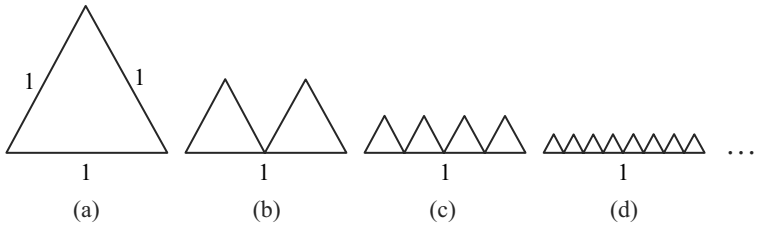


Figure 3.5

(a) At any stage the length of the zig-zag segment equals 2 units, because it is constructed from transformations of the original two sides of 1 unit each.

(b) On the other hand, from Figure 3.5 we can see as the number of stages n goes to infinity, the zig-zag curve gets closer and closer to the base of the triangle, which has length 1 unit. That is, $\lim_{n\to\infty} S_n = 1$, where S_n is the length of the zig-zag curve at stage n.

Comparing (a) and (b) we conclude that $2 = 1$.

15 Limit of perimeter curves shows $\pi = 2$.

Let us take a semicircle with diameter d. We divide the diameter into n equal parts and on each part construct semicircles of diameter $\frac{d}{n}$ as in Figure 3.6.

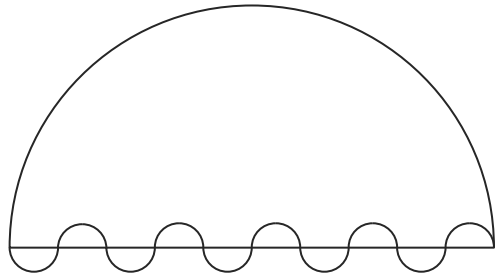


Figure 3.6

(a) The arc length of each small semicircle is $\frac{\pi d}{2n}$. The total length L_n of n semicircles is

$$L_n = \frac{\pi d}{2n} \times n = \frac{\pi d}{2}.$$

Therefore the limit of L_n when $n \to \infty$ is

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{\pi d}{2} = \frac{\pi d}{2}.$$

(b) From Figure 3.6 we can see that when n increases, the curve consisting of n small semicircles gets closer to the diameter, which has length d. That is $\lim_{n\to\infty} L_n = d$.

Comparing (a) and (b) we see that $\frac{\pi d}{2} = d$ and conclude that $\pi = 2$.

16 Serret's surface area definition proves that $\pi = \infty$.

Let us find the lateral surface area of a cylinder with height 1 unit and radius 1 unit using the following approach due to Serret. Divide the cylinder into n horizontal strips. Divide the circumference of each cross-section by points into m equal parts. Rotate the odd-numbered circumferences in such a way that the points on them are exactly midway between the points on the even-numbered circumferences. Form $2mn$ equal isosceles triangles by joining two adjacent points on each circumference with the point midway between them on the circumferences above and below. Two adjacent bands in this decomposition are in Figure 3.7.

First compute the area of one triangle, $\triangle PQR$, in the polyhedral surface. Here $\overline{PY} = 1/n$, and $\overline{OQ} = \overline{OR} = 1$. Segment PX denotes the altitude of triangle PQR. Thus the area of this triangle is $\frac{1}{2}\overline{QR} \times \overline{PX}$. Because the circumference is divided evenly into m portions, it follows that $\overline{QR} = 2 \sin \frac{\pi}{m}$.

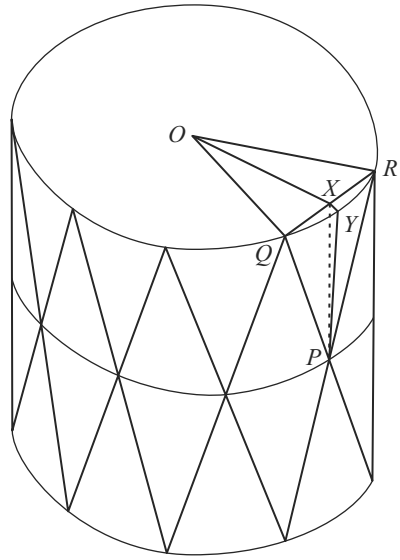


Figure 3.7

Now observe that $\triangle PYX$ is a right triangle with hypotenuse $\overline{PX}$ and leg $\overline{XY} = 1 - \cos\frac{\pi}{m}$. From the Pythagorean Theorem,

$$\overline{PX} = \frac{1}{n}\sqrt{1 + n^2\left(1 - \cos\frac{\pi}{m}\right)^2} = \frac{1}{n}\sqrt{1 + 4n^2\sin^4\frac{\pi}{2m}}.$$

The area of the polyhedral surface is thus given by

$$S_{mn} = 2m\sin\frac{\pi}{m}\sqrt{1 + 4n^2\sin^4\frac{\pi}{2m}}.$$

When both m and n tend to infinity this area tends to the lateral surface area of the cylinder. The limit of S_{mn} is found using the well-known formula $\lim_{x\to\infty}\frac{\sin x}{x} = 1$. Let us consider three cases.

(a) $n = m$:

$$\begin{aligned}\lim_{m\to\infty} S_m &= \lim_{m\to\infty} 2m\sin\frac{\pi}{m}\sqrt{1 + 4m^2\sin^4\frac{\pi}{2m}} \\ &= \lim_{m\to\infty} 2\pi\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\sqrt{1 + \frac{\pi^4}{4m^2}\left(\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^4} = 2\pi.\end{aligned}$$

(b) $n = m^2$:

$$\begin{aligned}\lim_{m\to\infty} S_m &= \lim_{m\to\infty} 2m \sin\frac{\pi}{m}\sqrt{1+4m^4\sin^4\frac{\pi}{2m}} \\ &= \lim_{m\to\infty} 2\pi \frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\sqrt{1+\frac{\pi^4}{4}\left(\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^4} \\ &= 2\pi\sqrt{1+\frac{\pi^4}{4}}.\end{aligned}$$

(c) $n = m^3$:

$$\begin{aligned}\lim_{m\to\infty} S_m &= \lim_{m\to\infty} 2m \sin\frac{\pi}{m}\sqrt{1+4m^6\sin^4\frac{\pi}{2m}} \\ &= \lim_{m\to\infty} 2\pi \frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\sqrt{1+\frac{\pi^4 m^2}{4}\left(\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^4} \\ &= \infty.\end{aligned}$$

In (a), (b) and (c) when $m \to \infty$ the polyhedral surface tends to the lateral surface of the cylinder. So the limit $\lim_{m\to\infty} S_m$ in (a), (b), and (c) must be the same.

This is possible only if $\pi = \infty$.

17 Achilles and the tortoise

This sophism, devised by the Greek philosopher Zeno in the 5th century B.C., is traditionally referred to as Zeno's paradox of Achilles and the tortoise or in short "The Achilles." With the definitions of "paradox" and "sophism" we use, Zeno's "paradoxes" are sophisms.

In a race between Achilles, the fastest of Greek warriors, and a tortoise, the tortoise has been granted a head start. Achilles will never pass the tortoise. Suppose the initial distance between the two is 1 unit and Achilles is moving 100 times faster than the tortoise. When Achilles covers the distance of 1 unit, the tortoise will have moved $\frac{1}{100}$th of a unit further from its starting point. When Achilles has covered the distance of $\frac{1}{100}$th of a unit, the tortoise will move $\frac{1}{100^2}$ of a unit further, and so on. The tortoise is always ahead of Achilles by $\frac{1}{100^n}$ of a unit no matter how long the race is. This means that Achilles will never reach the tortoise.

18 Reasonable estimations lead to $1{,}000{,}000 \approx 2{,}000{,}000$.

If we add 1 to a sufficiently large number the result would be approximately equal to the original number. For instance, if we take 1,000,000 and add 1 to it, we would agree that

$$1{,}000{,}000 \approx 1{,}000{,}001.$$

Similarly

$$1{,}000{,}001 \approx 1{,}000{,}002.$$

And

$$1{,}000{,}002 \approx 1{,}000{,}003.$$

And so on..., all the way to

$$1{,}999{,}999 \approx 2{,}000{,}000.$$

Multiplying the left-hand sides and the right-hand sides of the equalities we obtain

$$\begin{aligned} &1{,}000{,}000 \times 1{,}000{,}001 \times \cdots \times 1{,}999{,}999 \\ &\qquad \approx 1{,}000{,}001 \times 1{,}000{,}002 \times \cdots \times 2{,}000{,}000. \end{aligned}$$

Dividing both sides by $1{,}000{,}001 \times \cdots \times 1{,}999{,}999$ we conclude that

$$1{,}000{,}000 \approx 2{,}000{,}000.$$

19 Properties of square roots prove $1 = -1$.

Since $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, it follows that

$$1 = \sqrt{1} = \sqrt{(-1) \times (-1)} = \sqrt{-1} \times \sqrt{-1} = i \times i = i^2 = -1.$$

20 Analysis of square roots shows that $2 = -2$.

Two students were discussing square roots with their teacher.

The first student said, "A square root of 4 is -2".

The second student was skeptical, and wrote down $\sqrt{4} = 2$.

Their teacher commented, "You are both right".

The teacher was correct, so $2 = -2$.

21 Properties of exponents show that $3 = -3$.

Using the exponential rule

$$(a^m)^n = a^{mn}$$

With $a = -3$, $m = 2$, and $n = \frac{1}{2}$, we obtain

$$-3 = \left((-3)^2\right)^{\frac{1}{2}} = 9^{\frac{1}{2}} = \sqrt{9} = 3.$$

22 A slant asymptote proves that $2 = 1$.

Let us find the equation of a slant (or oblique) asymptote to the graph of the function

$$f(x) = \frac{x^2 + x + 4}{x - 1}$$

using two methods.

(a) Long division gives

$$f(x) = \frac{x^2 + x + 4}{x - 1} = x + 2 + \frac{6}{x - 1}.$$

The last term, $\frac{6}{x-1}$, tends to zero as $x \to \infty$. Therefore as $x \to \infty$ the graph of the function approaches the straight line $y = x + 2$, which is its slant asymptote.

(b) Dividing both numerator and denominator by x gives

$$f(x) = \frac{x^2 + x + 4}{x - 1} = \frac{x + 1 + \frac{4}{x}}{1 - \frac{1}{x}}.$$

Both $\frac{4}{x}$ and $\frac{1}{x}$ tend to zero as $x \to \infty$. Therefore as $x \to \infty$ the graph of the function approaches the straight line, $y = x + 1$, which is thus its slant asymptote.

Because

$$f(x) = \frac{x^2 + x + 4}{x - 1}$$

has only one slant asymptote, from (a) and (b) it follows that $x + 2 = x + 1$.

Subtracting x from both sides yields $2 = 1$.

23 Euler's interpretation of series shows $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$.

Euler began intensive investigations into series in the 1730s. Here is an argument he presented.

We know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

When we let $x = -1$ and evaluate the expressions on both sides, we obtain

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots .$$

24 Euler's manipulation of series proves $-1 > \infty > 1$.

Euler substituted $x = -1$ in

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

to get

$$\infty = 1 + 2 + 3 + 4 + \cdots .$$

Next he substituted $x = 2$ in

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

to get

$$-1 = 1 + 2 + 4 + 8 + \cdots .$$

Because $2^{n-1} \geq n$, for $n = 1, 2, \ldots$, we have

$$1 + 2 + 4 + 8 + \cdots > 1 + 2 + 3 + 4 + \cdots .$$

By the transitive property, it follows that $-1 > \infty$. Because $\infty > 1$, it follows that $-1 > \infty > 1$.

25 A continuous function with a jump discontinuity

For $k \geq 1$, define the function $f_k(x)$ on $[0, \infty)$ by

$$f_k(x) = \frac{1}{(k-1)x + 1} - \frac{1}{kx + 1}.$$

(a) Each $f_k(x)$ is continuous on $[0, \infty)$. If

$$S(x) = \sum_{k=1}^{\infty} f_k(x),$$

since each function $f_k(x)$ is continuous, the sum $S(x)$ is continuous at the values of x where the series converges.

(b) Let

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Because the terms form a telescoping sum we see that

$$S_n(x) = 1 - \frac{1}{nx+1}.$$

By definition, the series converges to $S(x)$ for the values of x where $\lim_{n\to\infty} S_n(x)$ exists. Because $S_n(0) = 0$, for all n, it follows $S(0) = 0$.

If $x > 0$, we obtain

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(1 - \frac{1}{nx+1}\right) = 1.$$

That is,

$$S(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}.$$

Comparing (a) and (b) it follows that the continuous function $S(x)$ has a jump discontinuity at $x = 0$.

26 Evaluation of Taylor series proves $\ln 2 = 0$.

Consider the Taylor series expansion for $\ln(1 + x)$,

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence is

$$\rho = \left(\lim_{n\to\infty} |a|^{1/n}\right)^{-1}.$$

Because $a_n = \frac{(-1)^{n+1}}{n}$, $\rho = 1$.

To determine the interval of convergence we check if the endpoints 1 and -1 are included in the interval of convergence. At $x = -1$, the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{-1}{n},$$

which diverges.

However at $x = 1$, the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n},$$

which converges by the alternating series test. Thus we conclude that the interval of convergence of the series is $(-1, 1]$.

Now since $x = 1$ lies in the interval of convergence, we have

$$\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

But

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\
&\quad = \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right) \\
&\quad = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right) \\
&\quad = 0.
\end{aligned}$$

Comparing our two computations, we conclude $\ln 2 = 0$.

4

Derivatives and Integrals

1 Trigonometric integration shows $1 = C$, for any real number C.

Let us apply the u-substitution method to find the indefinite integral $\int \sin x \cos x \, dx$ using two different substitutions:

(a) The substitution $u = \sin x$ with $du = \cos x \, dx$, gives

$$\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} + C_1 = \frac{\sin^2 x}{2} + C_1.$$

(b) The substitution $u = \cos x$ with $du = -\sin x \, dx$, gives

$$\int \sin x \cos x \, dx = -\int u \, du = -\frac{u^2}{2} + C_2 = -\frac{\cos^2 x}{2} + C_2,$$

where C_1 and C_2 are arbitrary constants. Equating the right-hand sides in (a) and (b) we obtain

$$\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$$

Multiplying by 2 and simplifying we obtain $\sin^2 x + \cos^2 x = 2C_2 - 2C_1$ or $\sin^2 x + \cos^2 x = C$, since the difference of two arbitrary constants is an arbitrary constant. On the other hand we know the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Therefore $1 = C$.

2 Integration by parts demonstrates $1 = 0$.

Let us find the indefinite integral $\int \frac{1}{x}\,dx$ using the formula for integration by parts $\int u\,dv = uv - \int v\,du$, with $u = \frac{1}{x}$ and $dv = dx$. This gives

$$\int \frac{1}{x}\,dx = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x}\,dx.$$

That is,

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Subtracting $\int \frac{1}{x}\,dx$ from both sides we conclude $0 = 1$.

3 Division by zero is possible.

Let us evaluate the indefinite integral $\int \frac{dx}{2x+1}$ by using the formula

$$\int \frac{f'(x)\,dx}{f(x)} = \ln|f(x)| + C$$

in two different ways.

(a) $$\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{dx}{x+\frac{1}{2}} = \frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_1$$

(b) $$\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{2\,dx}{2x+1} = \frac{1}{2}\ln|2x+1| + C_2.$$

By equating the right-hand sides in (a) and (b) it follows that

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_1 = \frac{1}{2}\ln|2x+1| + C_2.$$

Since C_1 and C_2 are arbitrary constants that can assume any values, let $C_1 = C_2 = 0$. Then

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right| = \frac{1}{2}\ln|2x+1|.$$

Solving for x we obtain $x+\frac{1}{2} = 2x+1$, so $x = -\frac{1}{2}$. Substituting this value of x into the original integral gives zero in the denominator, so division by zero is possible.

4 Integration proves $\sin^2 x = 1$ for any value of x.

Let us differentiate the function $y = \tan x$ twice:

$$y' = \frac{1}{\cos^2 x}, \quad y'' = \frac{2\sin x}{\cos^3 x}.$$

The second derivative can be written as

$$y'' = \frac{2\sin x}{\cos^3 x} = \frac{2\sin x}{\cos x \times \cos^2 x} = 2\tan x \times \frac{1}{\cos^2 x} = 2yy' = (y^2)'.$$

Integrating both sides of the equation $y'' = (y^2)'$ leads to

$$y' = y^2, \quad \text{or} \quad \frac{1}{\cos^2 x} = \tan^2 x, \quad \text{or} \quad \frac{1}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x}.$$

From here we conclude that $\sin^2 x = 1$, for any value of x.

5 The u-substitution method shows that $\frac{\pi}{2} < 0 < \pi$.

Let us estimate the integral $\displaystyle\int_0^\pi \frac{dx}{1+\cos^2 x}$.

(a) Since $\dfrac{1}{2} \le \dfrac{1}{1+\cos^2 x} \le 1$ on $[0, \pi]$, then

$$\int_0^\pi \frac{1}{2} dx \le \int_0^\pi \frac{dx}{1+\cos^2 x} \le \int_0^\pi dx$$

Or

$$\frac{\pi}{2} \le \int_0^\pi \frac{dx}{1+\cos^2 x} \le \pi.$$

(b) On the other hand, with $u = \tan x$ and $du = \frac{dx}{\cos^2 x}$, we have

$$\int_0^\pi \frac{dx}{1+\cos^2 x} = \int_0^\pi \frac{\frac{dx}{\cos^2 x}}{1+\tan^2 x+1} = \int_0^0 \frac{du}{2+u^2},$$

because $u = 0$ when $x = 0$ and $u = 0$ when $x = \pi$. Thus the integral has value 0.

From (a) and (b) we conclude that $0 \in \left[\frac{\pi}{2}, \pi\right]$.

6 $\ln 2$ is not defined.

Let us find the area enclosed by the graph of the function $y = \frac{1}{x}$, the x-axis, and the straight lines $x = -2$ and $x = -1$ using two different methods (see Figure 4.1).

(a) The derivative of $y = \ln x$ is $y' = \frac{1}{x}$ and therefore an antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln x$. We can apply the Newton-Leibniz formula to the integral $\int_{-2}^{-1} \frac{1}{x}\,dx$ (the limits are finite and the function is continuous on $[-2, -1]$) to find the required area:

$$A = -\int_{-2}^{-1} \frac{1}{x}\,dx = -\big(\ln(-1) - \ln(-2)\big).$$

This value is undefined since the logarithm of a negative number does not exist.



Figure 4.1

(b) On the other hand, the area is the same as the area enclosed by the graph of $y = \frac{1}{x}$, the x-axis, and the straight lines $x = 1$, and $x = 2$, due to the symmetry of the graph about the origin. Therefore the area equals

$$A = \int_{1}^{2} \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2.$$

Comparing (a) and (b) we conclude that $\ln 2$ is not defined.

7 π is not defined.

Let us find

$$\lim_{x\to\infty} \frac{\pi x + \sin x}{x + \sin x}$$

using two different methods.

(a) $\lim_{x\to\infty} \dfrac{\pi x + \sin x}{x + \sin x} = \lim_{x\to\infty} \dfrac{\pi + \frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = \pi.$

(b) Since both numerator and denominator are differentiable and approach ∞ as $x \to \infty$, we can use l'Hôpital's rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

which gives us

$$\lim_{x\to\infty} \frac{\pi x + \sin x}{x + \sin x} = \lim_{x\to\infty} \frac{\pi + \cos x}{1 + \cos x},$$

which is undefined.

From (a) and (b) we conclude that π is not defined.

8 Properties of indefinite integrals show $0 = C$, for any real number C.

We know that

$$\int k f(x)\, dx = k \int f(x)\, dx, \quad \text{where } k \text{ is a constant.}$$

Let us apply this property for $k = 0$.

(a) The left-hand side of the above equality becomes

$$\int 0 f(x)\, dx = \int 0\, dx = C, \quad \text{where } C \text{ is an arbitrary constant.}$$

(b) The right-hand side is $0 \int f(x)\, dx = 0$.

Comparing (a) and (b) we conclude that $0 = C$, where C is any real number.

9 Volumes of solids of revolution demonstrate that $1 = 2$.

Let us find the volume of the solid of revolution produced by rotating the hyperbola $y^2 = x^2 - 1$ about the x-axis on the interval $[-2, 2]$ using two different methods.

(a) $V = \pi \int_{-2}^{2} y^2\,dx = \pi \int_{-2}^{2} (x^2 - 1)\,dx = \pi \left(\frac{x^3}{3} - x \right) \Big|_{-2}^{2}$
$= \frac{4}{3}\pi$ (cubic units).

(b) Since the hyperbola is symmetric about the y-axis we can find the volume of a half of the solid of revolution, say on the right of the y-axis, and then multiply it by 2. Obviously the point $(1, 0)$ is the vertex to the right of the origin and the right branch of the hyperbola is to the right of the vertex $(1, 0)$. Therefore the volume of the right half is

$$V_1 = \pi \int_{1}^{2} y^2\,dx = \pi \int_{1}^{2} (x^2 - 1)\,dx = \pi \left(\frac{x^3}{3} - x \right) \Big|_{1}^{2}$$
$$= \frac{4}{3}\pi \quad \text{(cubic units)}$$

and the total volume is $V = 2V_1 = \frac{8}{3}\pi$ (cubic units).

Comparing (a) and (b) we deduce $\frac{4}{3}\pi = \frac{8}{3}\pi$ or $1 = 2$.

10 An infinitely fast fall

Imagine a cat sitting on the top of a ladder leaning against a wall. Suppose that the bottom of the ladder is being pulled away from the wall horizontally at a uniform rate. The cat speeds up, until it's eventually falling infinitely fast. The "proof" is below.

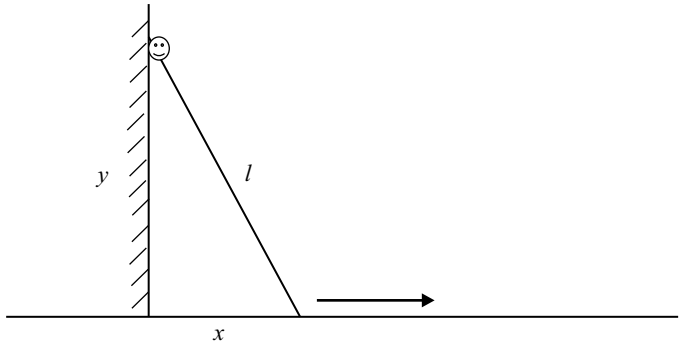


Figure 4.2

By the Pythagorean Theorem $y = \sqrt{l^2 - x^2}$, where $x = x(t)$, $y = y(t)$ are the horizontal and vertical distances from the ends of the ladder to

the corner at time t. Differentiation of both sides with respect to t gives us

$$y' = -\frac{xx'}{\sqrt{l^2 - x^2}}.$$

Since the ladder is pulled away at a uniform speed, x' is a constant. Let us find the limit of y' when x approaches l:

$$\lim_{x \to l} y' = \lim_{x \to l} \left(-\frac{xx'}{\sqrt{l^2 - x^2}} \right) = -\infty.$$

Therefore when the bottom of the ladder is pulled away by the distance l from the wall, the cat falls infinitely fast.

11 A positive number equals a negative number.

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(a) The function

$$f(x) = \frac{\sin x}{1 + \cos^2 x}$$

is continuous and nonnegative on the interval $\left[0, \frac{3\pi}{4}\right]$ and positive on any subinterval of the form $\left[a, \frac{3\pi}{4}\right]$, for $a > 0$. Therefore, by the definition of the definite integral, the area enclosed by the function $f(x)$ and the x-axis on the interval $\left[0, \frac{3\pi}{4}\right]$ is a positive number, or

$$\int_0^{\frac{3\pi}{4}} \frac{\sin x}{1 + \cos^2 x}\, dx > 0.$$

(b) That $F(x) = \tan^{-1}(\sec x)$ is an antiderivative of $f(x)$ is easy to check by differentiation. This allows us an alternative method to compute the definite integral:

$$\int_0^{\frac{3}{4}\pi} \frac{\sin x}{1 + \cos^2 x}\, dx = -\tan^{-1} \sqrt{2} - \frac{\pi}{4}.$$

Combining (a) and (b), we see that a positive number equals a negative number.

12 The power rule for differentiation proves that $2 = 1$.

Consider the following representation of x^2, for any $x \neq 0$.

$$x^2 = x + x + x + \cdots + x \quad (x \text{ copies}).$$

Differentiating both sides of the equation gives

$$2x = 1 + 1 + 1 + \cdots + 1 = x.$$

Division of both sides by x yields $2 = 1$.

Part III

Solutions to Paradoxes

5

Functions and Limits

1 Laying bricks

The top brick can extend infinitely far past the bottom brick! The x-coordinate of the position of the center of mass of a system of n objects with masses $m_1, m_2, \ldots, m_n$ is defined by the formula

$$x_0 = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}, \tag{5.1}$$

where x_i denotes the x-coordinate of the center of mass of the ith object. In our problem each of the bricks has an identical mass of m, and the center of mass for each brick lies at the geometric center of the brick.

First examine the case of two bricks. For the upper brick not to fall off the lower brick, the x-coordinate of the center of mass of the upper brick should not be positioned beyond the right edge of the lower brick. That is, the maximum value of the x-coordinate of the center of mass of the upper brick is l: $x_1 = l$. So the maximum shift is $s_1 = \frac{l}{2}$. (See Figure 5.1.)

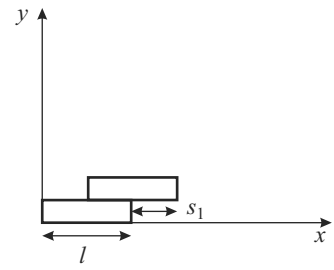


Figure 5.1

Next we consider three bricks in Figure 5.2.

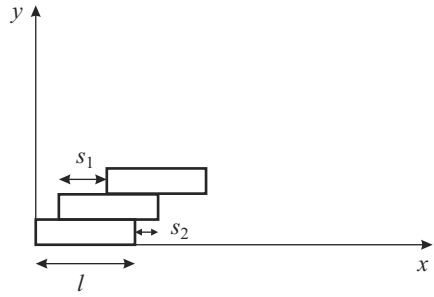


Figure 5.2

For the top brick we have already shown the maximum possible shift past the brick below it is $s_1 = \frac{l}{2}$. Let us find the maximum possible shift s_2 for the middle brick. This time we must ensure that the the x-coordinate of the center of mass of the system composed of the top two bricks does not exceed l.

From (5.1) we compute the center of mass and obtain

$$l = \frac{mx_1 + mx_2}{2m} = \frac{\left(s_2 + s_1 + \frac{l}{2}\right) + \left(s_2 + \frac{l}{2}\right)}{2}.$$

Solving this yields $s_2 = \frac{l}{4}$.

Continuing in this manner allows us to determine the sequence of maximum shifts, $s_3 = \frac{l}{6}, s_4 = \frac{l}{8}, \ldots, s_n = \frac{l}{2n}$.

The sum of the first n shifts is then

$$s_1 + s_2 + \cdots + s_n = \frac{l}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

It is well known that the harmonic series diverges, so when $n \to \infty$ the sum in the parentheses tends to infinity. This means that the top brick can extend as far past the bottom brick as we want.

In practice it is, of course, impossible to build such an arch. Starting from a certain value of n we will not be able to make shifts of the length $\frac{1}{2n}$ as they will be too small to perform.

Martin Gardner brought this problem to popular attention with his *Scientific American* article, "Some paradoxes and puzzles involving infinite series and the concept of limit" [11]. He refers to it as the "infinite-offset paradox." Interested readers can also delve more deeply into this question with recent *American Mathematical Monthly* articles focused on the maximum overhang of bricks problem [26] and [27].

2 Spiral curves

(a) Let us construct a spiral curve of a finite length.

First draw a line segment of length d. Draw a semicircle with diameter d on one side of the line segment. Then on the other side of the line segment draw a semicircle of diameter $d/2$. Then on the other side draw a semicircle of diameter $d/4$, and so on, as in Figure 5.3.

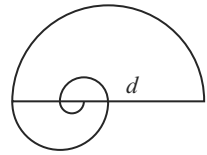


Figure 5.3

The length of the curve is given by the geometric series

$$\pi\frac{d}{2}+\pi\frac{d}{4}+\pi\frac{d}{8}+\cdots=\pi d\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)=\pi d.$$

(b) Now let us construct a spiral curve of infinite length. Draw a line segment AB of length d with midpoint P. Choose a value of a, with $0 < a < d/4$. Let C be a point on AB that is at distance a from P. Draw a circle with the center C and radius a. Let D denote the other endpoint of the diameter PD for this circle. On one side of the line segment AB draw a semicircle of diameter d. On the other side of the line segment draw a semicircle of diameter AE, where point E is the midpoint of PB. Then on the other side of the line segment draw a semicircle of diameter EF, where point F is the midpoint of AD and so on (see Figure 5.4).

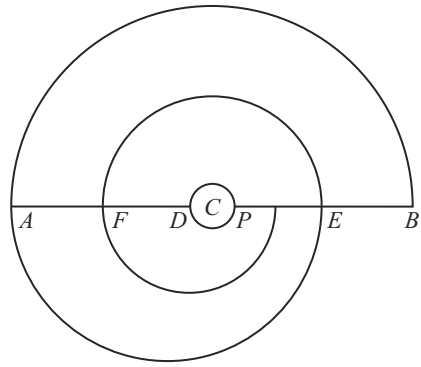


Figure 5.4

The curve has infinitely many rotations around C and each rotation has a length bigger than the circumference $2\pi a$, so the length of the curve is infinite.

Readers interested in spirals may enjoy learning more about the logarithmic spiral (also called the equiangular spiral) associated with the golden mean or the Archimedean spiral. For a quick introduction to either topic, consider Pappas' *The Joy of Mathematics* [24, pp. 105, 149, 189], or *More Joy of Mathematics* [25, pp. 136, 146, 147]. A more in-depth study of the logarithmic spiral can be found in Maor's *e: The Story of a Number* [19, pp. 121–127] and more coverage of spirals in general is given by Gazalé in Chapter 8 of *Gnomon: From Pharoahs to Fractals* [12].

3 A paradoxical fractal curve: the Koch snowflake.

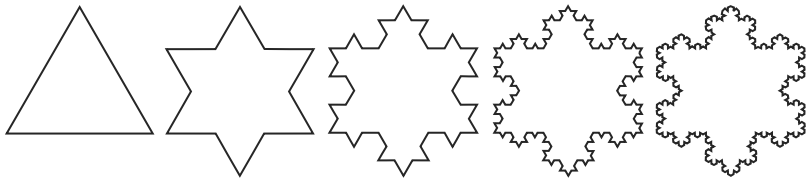


Figure 5.5

The initial triangle and all consecutive stars and snowflakes in Figure 5.5 lie between the incircle and the circumcircle of the original triangle. These circles have circumferences of finite length.

Now we examine the perimeter of the star curves. If the perimeter of the initial triangle is 1 unit, then the perimeter of the star in the first iteration is $12 \times \frac{1}{9} = \frac{4}{3}$ units. The perimeter of the snowflake in the second iteration is $48 \times \frac{1}{27} = \frac{16}{9} = \left(\frac{4}{3}\right)^2$ units. The perimeter of the snowflake in the nth iteration is $\left(\frac{4}{3}\right)^n$ units. As $n \to \infty$ the perimeter of the snowflake tends to infinity. The Koch curve has an infinite length while it bounds a finite area! Note the area of the snowflake is less than the area of the circumcircle minus the area of the incircle.

Reference materials abound for further study of the Koch snowflake curve and related fractals. We first mention two books that provide associated lesson plans for teachers: *Fractals for the Classroom: Strategic Activities*, an NCTM aligned resource, [28], and *Fractals: A Tool Kit of Dynamics*

Activities, [7]. Readers seeking more information on the intriguing nature of this and other strange curves, may enjoy *Curious Curves*, [9].

4 A tricky fractal area: the Sierpinski carpet

Although at every step the remaining area is 8 times bigger than the area removed, the limit of the remaining area will be zero and thus the limit of the total area removed will be 1 square unit. Let us see why this is the case. After the first step, the remaining area equals $1 - \frac{1}{9} = \frac{8}{9}$. After the second step, the remaining area is

$$\frac{8}{9} - 8 \times \frac{1}{81} = \frac{64}{81} = \left(\frac{8}{9}\right)^2.$$

Similarly, after the nth step the remaining area is $\left(\frac{8}{9}\right)^n$. As n tends to infinity this area tends to zero. For references on the Sierpinski carpet see the materials mentioned in the answer to Paradox 3.

5 A mysterious fractal set: the Cantor ternary set

First compute the size of $C(n)$. At stage n, $C(n)$ consists of 2^n disjoint closed intervals each of length $(\frac{1}{3})^n$. Thus the proportion of the original interval that remains at stage n is $2^n/3^n$. Clearly

$$\lim_{n\to\infty} \frac{2^n}{3^n} = 0.$$

That is, the Cantor ternary set C accounts proportionally for 0% of the original interval.

Now we examine the ternary expansion of the points in the Cantor ternary set. Here a value $(0.a_1a_2a_3\ldots a_n)_3$ in ternary stands for the number

$$\sum_{i=1}^{n} \frac{a_i}{3^i},$$

where $a_i = 0$, 1, or 2. For example, $.2_3 = \frac{2}{3}$ and $.201_3 = \frac{2}{3} + \frac{1}{27}$. Ternary expansions need not terminate. A value $(0.a_1a_2a_3\ldots a_n\ldots)_3$ represents

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

For instance

$$.\overline{1}_3 = \sum_{i=1}^{\infty} \frac{1}{3^i} = \frac{1}{2}.$$

As in the decimal system, any number with a terminating ternary expansion can also be represented with a non-terminating expansion. For example, $\frac{1}{9} = .01_3$ can also be written as

$$\frac{1}{9} = .00\overline{2}_3 = \sum_{i=3}^{\infty} \frac{2}{3^i}.$$

This observation proves particularly helpful when we describe the points which remain in the Cantor ternary set. At stage 1 remove $\left(\frac{1}{3}, \frac{2}{3}\right)$ and what remains is

$$C(1) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Observe that $[0, \frac{1}{3}]$ consists of all ternary numbers between $0 = .0_3$ to $\frac{1}{3} = .0\overline{2}_3$. In other words, $[0, \frac{1}{3}]$ is the set of numbers in $[0, 1]$ that have a ternary expansion containing $a_1 = 0$. Similarly $[\frac{2}{3}, 1]$ consists of all ternary numbers between $\frac{2}{3} = .2_3$ to $1 = .\overline{2}_3$, or all numbers in $[0, 1]$ that have a ternary expansion with $a_1 = 2$. Thus at stage 1, we exclude values whose representations in ternary form require $a_1 = 1$. Continuing at stage 2, where $C(2) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, note that the endpoints have the following ternary expansions:

$$0 = .0_3,\ \frac{1}{9} = .00\overline{2}_3,\ \frac{2}{9} = .02_3,\ \frac{1}{3} = .0\overline{2}_3,$$
$$\frac{2}{3} = .2_3,\ \frac{7}{9} = .20\overline{2}_3,\ \frac{8}{9} = .22_3,\ \text{and}\ 1 = .\overline{2}_3.$$

It follows that the points which remain in $C(2)$ are those that possess a ternary expansion with both $a_1 = 0$ or 2 and $a_2 = 0$ or 2. Continuing on in this manner we realize the values that remain in the Cantor ternary set are those which admit a ternary expansion consisting entirely of 0s and 2s. Now associate with each point $(0.a_1a_2a_3 \ldots a_n \ldots)_3$ in C the corresponding point in $[0, 1]$ with binary expansion $(0.b_1b_2b_3 \ldots b_n \ldots)_2$, for $b_i = \frac{a_i}{2}$. The points $(0.b_1b_2b_3 \ldots b_n \ldots)_2$ provide all possible binary expansions for points in $[0, 1]$. Thus the Cantor ternary set contains as many points as our original interval!

6 A misleading sequence

(a) While the expected answer is 32, there are infinitely many other correct answers. In fact, the next term a_5 in the sequence 2, 4, 8, 16 can take on any value a.

Consider the sequence rule for the nth term given by

$$a_n = 2^n + (n-1)(n-2)(n-3)(n-4)x.$$

The first four terms of this sequence are 2, 4, 8, 16. We can make $a_5 = a$ by taking $x = \frac{(a-32)}{24}$.

(b) Another interesting combinatoric example from geometry produces the sequence 2, 4, 8, 16 for the first four terms, but surprises us with its fifth term. Draw a circle, put two dots on the circumference and connect them with a line segment. The circle is divided into two regions. Put a third dot and connect all dots. The circle is now divided into four regions. Put a fourth dot and connect all dots. The circle is now divided into eight regions. (See Figure 5.6) Put a fifth dot and connect all dots. The circle is now divided into sixteen regions. It looks like we have a clear pattern. But when you put a sixth dot and connect all dots the circle is divided into thirty regions!

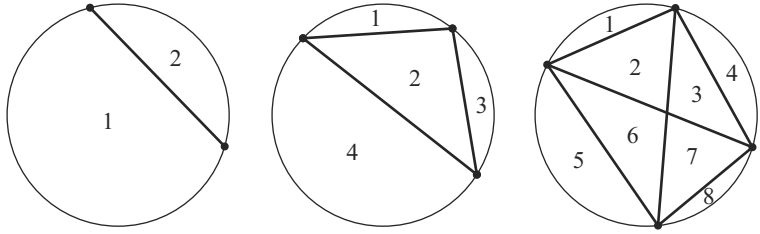


Figure 5.6

7 Remarkable symmetry: Reuleaux polygons

No, the student was not right. A figure can be of constant diameter and yet not be a circle. As an example, consider the following curve. In an equilateral triangle draw circular arcs with the radius equal to the side of the triangle from each vertex. The resulting curved triangle, known as the Reuleaux triangle, (see Figure 5.7) has a constant diameter. When it rolls on a horizontal surface its center moves along a sine curve, undulating up and down

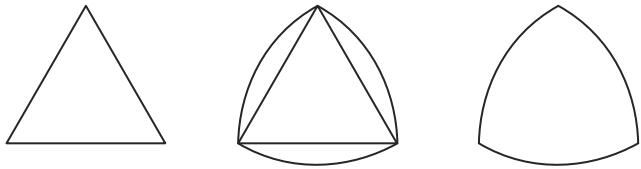


Figure 5.7

(the center of a rolling circle does not move up and down). For this reason it is not practical to use it as a wheel. However, the Reuleaux triangle does have practical applications. Wankel rotary engines employ Reuleaux triangles. And some cities distinguish potable water from waste water by utilizing Reuleaux triangular covers for one type of valves and circular covers for the others.

For any odd number n, this construction can be modified to produce a Reuleaux n-gon. We begin with an n-pointed star inscribed in a circle. First evenly space n points on the circle. Then connect every $m = (n-1)/2$ points on the circle to form a star. Each vertex of the star is connected to exactly two other vertices and these two vertices are adjacent on the circle. From each vertex, draw circular arcs with radius equal to the side length of the star. These arcs join pairs of adjacent vertices on the circle. The Reuleaux n-gon results. In Figure 5.8 is a five-pointed star and the resulting Reuleaux pentagon.

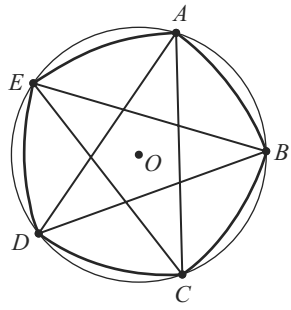


Figure 5.8

Reuleaux heptagons (7-gons) have often been used for British coinage, including two pence, twenty pence, and fifty pence pieces. Their constant diameter allows for their use in coin-operated vending machines. The Reuleaux heptagon is of smaller area than the associated circle of the same radius, so a Reuleaux coin requires less material.

Chapter 10 of *How Round Is Your Circle?*, [4] offers an engaging chapter on Reuleaux polygons in addition to solids of constant width. A number

of interesting physical and engineering applications are discussed there as well.

8 Rolling a barrel

The barrel rolls as long as the person continues walking. The velocity of the point on the top of the barrel equals the velocity of the walking person. The velocity of any point on the cylinder is given by the sum of the rotational and translational velocity. Here the translational velocity is the velocity of the axis of the barrel and the rotational velocity is the velocity of the point on the barrel with respect to the axis. For the point on the top of the barrel, both the translational and rotational velocities are $R\omega$, where R is the radius of the barrel and ω is the angular velocity. In other words, the velocity of the point on top of the barrel is twice the velocity of the axis of the barrel. (Perhaps even more strangely, the velocity of the point on the bottom of the barrel is always 0!) Thus, the person will cover twice the length of the board, or 6 m by the time he reaches the barrel.

9 A cat on a ladder

Part 1. Most people confidently choose the correct answer C to Part 1. Without much difficulty, one can imagine the ladder rotating about a central point, i.e., where the base of the ladder touches the wall. A circular arc then results as in Figure 5.9.

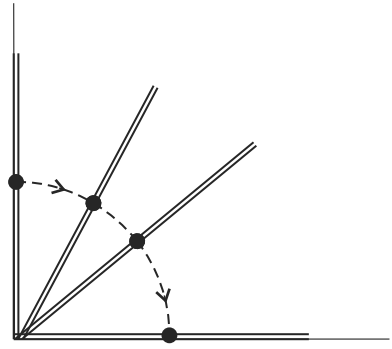


Figure 5.9

Part 2. However, Part 2 isn't so easy to solve intuitively! Many people are convinced that A is the correct answer. They may reason that as the ladder slides away from the wall that it would appear to drop quickly, then level out as it approaches the horizontal. Surprisingly, the answer to this

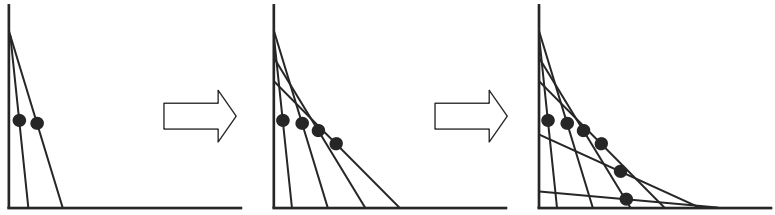


Figure 5.10

problem is also C. To gain a sense of what is happening, you may want to first try making a model for this problem. (See Figure 5.10.) Mark the midpoint of a paper ladder. Slowly slide the ladder down and away from the wall, always keeping the endpoints of the ladder on the wall and floor. After each movement, put a dot on the page where the center of the ladder lies. As the ladder approaches the horizontal, further lateral movement is minimal.

Here is a simple coordinate geometry proof for the situation in part 2. Let AB be the ladder and label the midpoint of AB as C, in honor of the cat. (See Figure 5.11.)

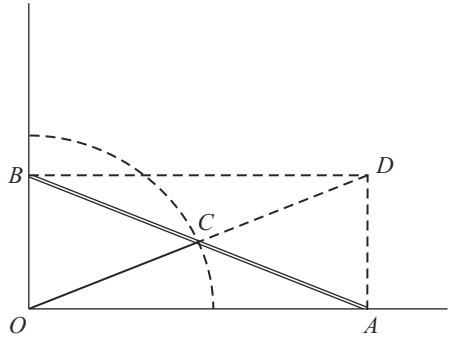


Figure 5.11

Suppose that $A = (a, 0)$ and $B = (0, b)$. This implies the coordinates of the cat are at $C = (a/2, b/2)$. Observe that as the ladder moves, points A and B move along the x and y-axes respectively, but the length of $AB = \sqrt{a^2 + b^2}$ remains fixed. The point C always lies on circle of radius $AB/2$ centered at the origin. That is C always lies on the circle

$$x^2 + y^2 = \frac{a^2 + b^2}{4}.$$

You may well be surprised to learn the trajectory is the same in either case. However, intuition fails many people in this pair of problems. A colleague tested groups of 100 fourth-year engineering students in Australia,

Germany, New Zealand, and Norway. Students of engineering are expected to be able to quickly conceptualize shapes, movements, and forces. They were given forty seconds for this mental exercise with no calculations or drawings permitted. The results were startling, for although 74% of the students gave C, the correct answer to Part 1, 86% selected the wrong response in Part 2 (52% chose A and 34% choose B).

Part 2 of this problem is a special case of the trammel of Archimedes problem. There, the cat need not sit at the midpoint, but can be placed at any point on the ladder. As the ladder slides with its endpoints A and B remaining on the x and y-axes, the trajectory of the cat is part of an ellipse. Devices constructed to model this are called ellipsographs. Interested readers can learn more about this subject from a recent article [1] "A New Look at the So-Called Trammel of Archimedes."

10 Sailing

The top of the yacht has covered the longest distance. The shape of the Earth is approximately spherical, so the top of the yacht has the longest radius compared to lower parts and therefore has the longest circumference.

11 Encircling the Earth

The correct answer of approximately 3 m high is a surprise to many people. Let r be the radius of the Earth and R be the radius of the circle after adding 20 meters to the rope. The difference between the two circumferences is 20 m: $2\pi R - 2\pi r = 20$ or $2\pi(R - r) = 20$. From here it follows that the difference between the two radii is $R - r = 10/\pi \approx 3$ m. The answer does not depend on the original length of the rope! Thus if we replaced the Earth with a mere basketball, the result would be the same.

12 A tricky equation

The rough sketches of the graphs are too rough. The graphs of $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$ are very close to each other on the interval $(0.1, 0.7)$ and, in fact, they intersect three times on this interval. If we use a computer to draw the correct graphs and zoom them we can see that the equation has three solutions: 0.25, ≈ 0.364, and 0.5, See Figures 5.12–5.15. The first is on the interval $(0.1, 0.7)$ and the other three on the intervals around the solutions 0.25, ≈ 0.364, and 0.5.

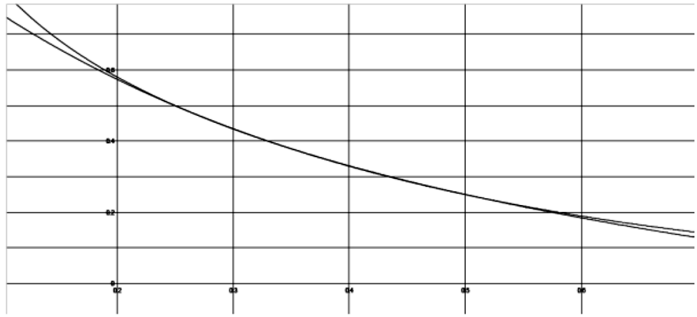


Figure 5.12 The graphs of the functions $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$ on $(0.1, 0.7)$

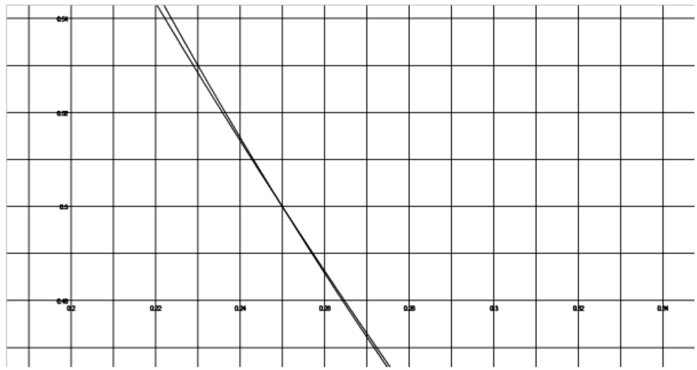


Figure 5.13 The graphs of the functions $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$ on $(0.22, 0.34)$

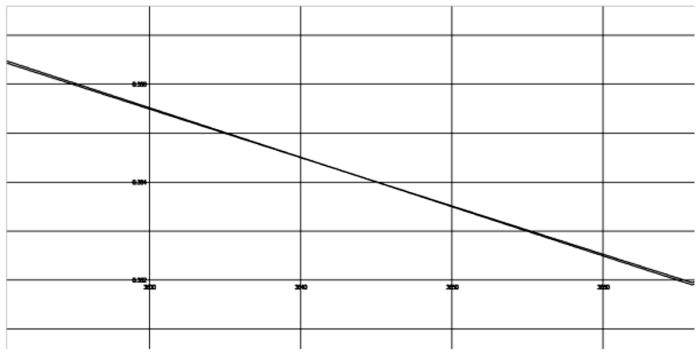


Figure 5.14 The graphs of the functions $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$ on $(0.362, 0.366)$

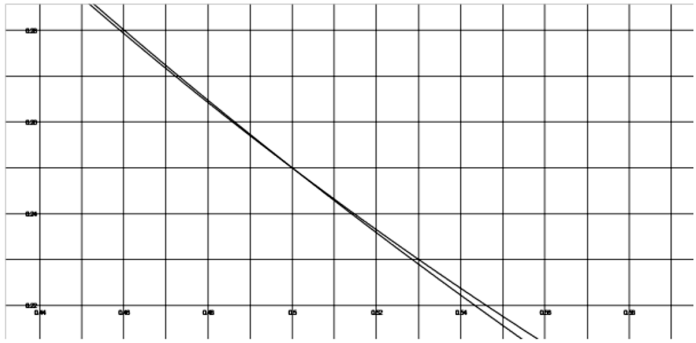


Figure 5.15 The graphs of the functions $y = \log_{1/16} x$ and $y = \left(\frac{1}{16}\right)^x$ on $(0.44, 0.56)$

If we use the solve command in Matlab to solve the equation $\log_{1/16} x = \left(\frac{1}{16}\right)^x$ we receive only one solution, 0.5.

13 A snail on a rubber rope

Yes, the snail will reach the end of the rope. Let us denote by a_n the portion of the rope the snail has covered just before the stretching at time n. In the first minute, the snail advances 1 cm by his own efforts. Thus he has covered $\frac{1}{100}$th $= a_1$ of the rope in this time frame. Upon the first stretching of the rope uniformly by 1 m, we have a scaling factor of 2, so the snail is carried forth by this elongation to the 2 cm position. In the second minute the snail moves by himself another 1 cm. Thus his new position is at the 3 cm mark, and we have $3\text{ cm} = \left(\frac{1}{100} + \frac{1}{200}\right) 2\text{ m}$, so that $a_2 = \frac{1}{100}\left(1 + \frac{1}{2}\right)$. The second stretching by 1 m induces a scalar factor of $3/2$, so the snail is now pulled along to a position $\frac{9}{2}$ cm away from his initial starting point. In the third minute he advances 1 cm again under his own power. His position before the third stretching is at the $11/2$ cm mark. Observe that

$$\frac{11}{2}\text{ cm} = \left(\frac{1}{100} + \frac{1}{200} + \frac{1}{300}\right) 3\text{ m},$$

or that $a_3 = \frac{1}{100}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$. Continuing inductively, we can show that the fraction of the rope traversed at time n before the stretching takes place is $a_n = \frac{1}{100}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$. Because the harmonic series diverges, we know that for some value of n, $\sum_1^n \frac{1}{i} > 100$, or that at this stage $a_n > 1$. It follows that the snail will eventually reach the end of the rope.

6

Derivatives and Integrals

1 An alternative product rule

The "product rule" is true when u and v are any functions that satisfy the differential equation $u'v' = u'v + uv'$. We can let u be any function and seek v.

For example, suppose we let $u = x$. Then we need v to satisfy $v' = v + xv'$. We can solve this system for v by separation of variables:

$$\begin{aligned} v'(1-x) &= v, \\ \frac{dv}{v} &= \frac{dx}{1-x}, \\ \ln|v| &= -\ln|1-x| + \ln|c|, \\ \ln|v| &= \ln\left|\frac{c}{1-x}\right|, \\ v &= \frac{c}{1-x}, \end{aligned}$$

where c is an arbitrary constant. In particular,

$$u = x \qquad \text{and} \qquad v = \frac{1}{1-x}$$

serve as a solution pair.

As a result, there are infinitely many pairs of functions for which the "rule" is true.

2 Missing information?

Let R be the radius of the sphere. The radius of the drilled hole is then $r = \sqrt{R^2 - 64}$. The volume that remains once the hole has been drilled equals the volume of the solid of revolution when the shaded area rotates around the x-axis (see Figure 6.1):

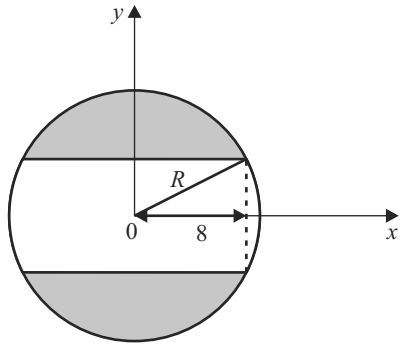


Figure 6.1

$$V = 2\pi \int_0^8 (R^2 - x^2 - r^2)\,dx = 2\pi \int_0^8 (R^2 - r^2 - x^2)\,dx$$
$$= 2\pi \int_0^8 (64 - x^2)\,dx = \frac{2048\pi}{3}\ \text{cm}^3.$$

3 A paint shortage

This solid of revolution known as Torricelli's trumpet or Gabriel's horn presents one of the most famous paradoxes to involve areas and volumes. The general approach to explain paradoxes that involve infinity and physical objects (like paint in this case) is to differentiate the "mathematical" universe from the "physical" universe. Infinity is a pure mathematical idea that we cannot apply to real, finite objects. From a mathematical point of view one "abstract" drop of paint is enough to cover any area, no matter how large. One just needs to make the thickness of the cover very thin, and infinitely thin if you want to cover an infinite area. Consider an easier example. You have 1 drop of paint which has the volume of 1 cubic unit. You need to cover a square plate of the size x by x units. Then the (uniform) thickness of the cover will be $\frac{1}{x^2}$ units. If $x = 100$ cm, then the thickness of the cover is $\frac{1}{10000}$ cm. If $x \to \infty$, then the area $x^2 \to \infty$ and the thickness

$\frac{1}{x^2} \to 0$. But at any stage the volume is

$$x^2 \times \frac{1}{x^2} = 1 \text{ cubic cm.}$$

So mathematically you can cover any infinite area with any finite amount of paint, even with a single drop. In reality such infinite areas don't exist, nor can one make the cover infinitely thin. For further historical background and more in-depth study of Torricelli's trumpet, see Chapter 8 of *Nonplussed* by Havil [13].

4 Racing marbles

Our curve is given by the parametric equations $x(\theta) = \theta - \sin\theta$ and $y(\theta) = \cos\theta - 1$, for $0 \le \theta \le \pi$. Let v denote the velocity of the marble at any point (x, y) along its path. We have $v = \frac{ds}{dt}$, where $ds = \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta$. We compute ds along the curve and have

$$ds = \sqrt{(1-\cos\theta)^2 + (-\sin\theta)^2}\,d\theta = \sqrt{2 - 2\cos\theta}\,d\theta = \sqrt{-2y}\,d\theta.$$

We first compute the amount of time it would take a marble that starts at the origin to travel to the endpoint $(\pi, -2)$. Let m denote the mass of the marble. Then by conservation of energy, we have

$$\frac{1}{2}mv^2 = -mgy.$$

Thus

$$v = \frac{ds}{dt} = \sqrt{-2gy}, \quad \text{and} \quad dt = \frac{ds}{\sqrt{-2gy}} = \frac{d\theta}{\sqrt{g}}.$$

Thus the amount of time it takes the marble to travel from the origin to the bottom point is

$$T = \frac{1}{\sqrt{g}} \int_0^\pi d\theta = \frac{\pi}{\sqrt{g}}.$$

Suppose instead that the marble starts from an intermediate point $M = \big(x(\alpha), y(\alpha)\big)$, with $0 < \alpha < \pi$. This time conservation of energy gives

$$\frac{1}{2}mv^2 = mg\big(y(\alpha) - y\big)$$

Or that

$$dt = \frac{ds}{\sqrt{2g(\cos\alpha - \cos\theta)}} = \frac{1}{\sqrt{g}}\sqrt{\frac{1-\cos\theta}{\cos\alpha - \cos\theta}}\,d\theta.$$

To compute the time it takes to travel from M to the end of the track,

$$T = \frac{1}{\sqrt{g}} \int_{\alpha}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\alpha - \cos\theta}}\, d\theta,$$

first employ the identities $1-\cos\theta = 2\sin^2(\frac{\theta}{2})$ and $1+\cos\theta = 2\cos^2(\frac{\theta}{2})$.

Then we have

$$\begin{aligned} T &= \frac{1}{\sqrt{g}} \int_{\alpha}^{\pi} \sqrt{\frac{2\sin^2(\theta/2)}{(1+\cos\alpha)-(1+\cos\theta)}}\, d\theta \\ &= \frac{1}{\sqrt{g}} \int_{\alpha}^{\pi} \frac{\sin(\theta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\theta/2)}}\, d\theta. \end{aligned}$$

Next utilize the trigonometric substitution with $u = \frac{\cos(\theta/2)}{\cos(\alpha/2)}$. The bounds $\theta = \alpha$ and $\theta = \pi$ correspond to the bounds $u = 1$ and $u = 0$ respectively, and $du = -\frac{\sin(\theta/2)d\theta}{2\cos(\alpha/2)}$. Rewriting the integral then gives

$$T = \frac{1}{\sqrt{g}} \int_{0}^{1} \frac{2du}{\sqrt{1-u^2}} = \frac{\pi}{\sqrt{g}}.$$

Thus, the amount of time the marble needs to roll to the finish is independent of the starting point M.

The curve described by this pair of parmetric equations is a cycloid. The problem of finding a curve for which a marble placed anywhere will reach the bottom in the same amount of time is classically referred to as the tautochrone problem. Huygens first discovered and published a solution. Cycloids have been studied extensively and exhibit a number of other intriguing properties as well. Readers may enjoy the well written Chapter 9 on cycloids in [18].

5 A paradoxical pair of functions

First we verify that both $f(x)$ and $g(x)$ are continuous at $x = 0$. In each case the squeeze theorem for limits applies. We are naturally able to find upper and lower bound functions for $f(x)$ and $g(x)$:

$$-|x| \le x\sin\frac{1}{x} \le |x|,$$

and

$$-x^2 \le x^2\sin\frac{1}{x} \le x^2.$$

It follows that both $\lim_{x\to 0} x \sin\frac{1}{x} = 0$ and $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ or that $f(x)$ and $g(x)$ are continuous at $x = 0$.

The distinction in the differentiability of the two functions arises when we examine the difference quotients in the definition of the derivative. To see that $f(x)$ is not differentiable at $x = 0$, observe that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x}$$

does not exist.

On the other hand, we can demonstrate that $g(x)$ is differentiable at $x = 0$. We have

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x} \\ &= \lim_{x\to 0} x \sin\frac{1}{x} = 0. \end{aligned}$$

6 An unruly function

Yes, such functions exist!

To showcase this possibility, we present a classical example that relies on knowing that both the set of rational numbers and the set of irrational numbers form dense subsets of the real numbers. The rationals are dense in the real number line because for any two distinct real numbers a and b, with $a < b$, we can always find a rational number r such that $a < r < b$. A similar statement holds for the density of the irrationals.

Now employ the clever idea of defining a function $f(x)$ based on the rationality or irrationality of x. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We first show that $f(x)$ is not continuous on any interval (a, b). In fact, $f(x)$ is continuous only at the single point $x = 0$.

By definition a function $f(x)$ is continuous at $x = a$ if given any $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$, this guarantees $|f(x) - f(a)| < \epsilon$. Thus to prove that $f(x)$ is not continuous at $x = a$, we need only exhibit a value of $\epsilon > 0$ for which no matter what $\delta > 0$ we take, we can always produce a value of x where $|x - a| < \delta$ holds and yet $|f(x) - f(a)| \geq \epsilon$.

We demonstrate that $f(x)$ is not continuous at any rational value $a \neq 0$. (It can also be shown that $f(x)$ is not continuous at any irrational value a.)

Once this is established, it is an immediate consequence of the density of the rationals that $f(x)$ is not continuous on any interval (a, b).

Suppose that $a \neq 0$ is a rational number, so that $f(a) = a^2 \neq 0$. Let $\epsilon = \frac{a^2}{2}$. Now consider any $\delta > 0$. By the density of the irrationals, there exists an irrational number x such that $x \in (a - \delta, a + \delta)$. At this value of x, we have $f(x) = 0$, $|x - a| < \delta$, and $|f(x) - f(a)| = a^2 > \epsilon$. Thus $f(x)$ is not continuous at $x = a$.

Remarkably, $f(x)$ is both continuous and differentiable at $x = 0$. To see this we verify the respective limit definitions.

For the continuity of $f(x)$ at $x = 0$, consider any $\epsilon > 0$. Let $\delta = \sqrt{\epsilon}$. We then have $|f(x) - f(0)| < \epsilon$, whenever $|x - 0| < \delta = \sqrt{\epsilon}$.

As for the differentiability of $f(x)$ at $x = 0$, compute

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{f(x)}{x}. \end{aligned}$$

We can utilize the squeeze theorem to evaluate this limit because $0 \leq f(x) \leq x^2$, so $0 \leq \left|\frac{f(x)}{x}\right| \leq |x|$. We conclude that $f'(0)$ exists and $f'(0) = 0$.

7 Jagged peaks galore

In the 1830s Bolzano produced the first known example of a continuous function that is not differentiable anywhere. However, his discovery did not initially draw much attention and his manuscript was not published until roughly one hundred years later.

Nitecki [21, pp. 237–246], presents a detailed geometric construction of a continuous nowhere differentiable function on $[0, 1]$. The inductive construction first defines a sequence of piecewise linear functions with specified values at the triadic rationals. (Triadic rationals are those points with terminating ternary expansions $0.a_1a_2a_3 \ldots a_n$. See Paradox #5: The Cantor ternary set for more details about such expansions.) The function is then extended to all real values in $[0, 1]$ by taking limits of these functions and utilizing the definition of continuity.

Nitecki also discusses the intriguing historical events surrounding this problem. Based on an understanding of the exposition on the triadic rationals construction, he leads readers through a series of exercises that explore Bolzano's own construction. This treatment is accessible to advanced first-

year calculus students and offers wonderful material for exploration or an honors project.

The first example of a continuous nowhere differentiable function is attributed to Weierstrass. In the 1870s he presented his trigonometric series based result. The function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos(3^n x)$$

represents a special case of his example. Figure 6.2 shows the first seven terms in this series and allows us to visualize its fractal nature.

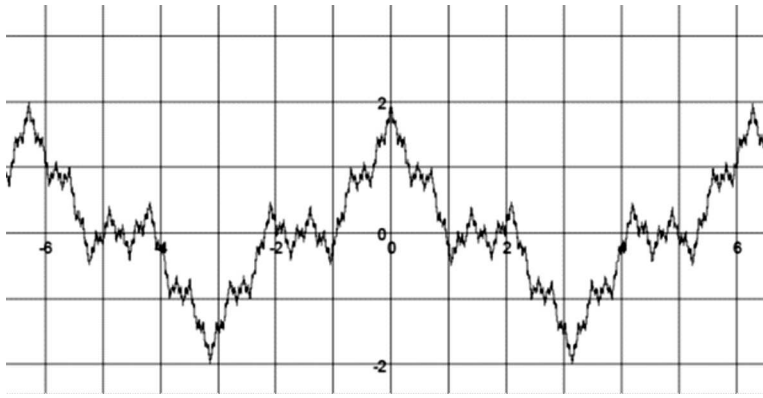


Figure 6.2

The proof of the continuity and nowhere differentiability of the Weierstrass function requires more advanced analysis tools and techniques. For those familiar with Fourier series and uniform convergence, we recommend the coverage in [3, pp. 259–263].

These fractal examples of Bolzano and Weierstrass oscillate at every level. One could reasonably wonder, how extreme can the behavior of a monotone continuous function be? After developing his continuous nowhere differentiable function, Weierstrass went on to explore this question. He constructed a monotone continuous function that is not differentiable at any rational number. Weierstrass futher believed that he could construct a monotone continuous nowhere differentiable function; he kept on searching for an example. However, Lebesgue later proved him wrong by demonstrating that no such function could exist! While we can understand the question Weierstrass pursued in the context of first-year calculus, the result of Lebesgue lies in the realm of a much more advanced area of analysis called measure theory.

8 Another paradoxical pair of functions

We first comment on the discontinuous nature of $f(x)$ and $g(x)$. While the function $f(x)$ is, in fact, discontinuous for every value of x, we confirm only its discontinuity for rational points. Take any rational value $x \in [0, 1]$ and let $\epsilon = \frac{1}{2}$. Given any $\delta > 0$, we can find an irrational value $y \in (x - \delta, x + \delta)$ by the density of the irrational numbers. At y we have

$$\big|f(x) - f(y)\big| = |1 - 0| = 1 > \epsilon = \frac{1}{2}, \text{ while } |x - y| < \delta.$$

It follows that f is discontinuous at x.

We show that the function $g(x)$ is discontinuous at every rational x in a similar manner. For $x = \frac{p}{q}$, in lowest terms, consider the value $\epsilon = \frac{1}{2q}$. For any $\delta > 0$, we can again find an irrational value $y \in (x - \delta, x + \delta)$. There

$$\big|g(x) - g(y)\big| = \left|\frac{1}{q} - 0\right| = \frac{1}{q} > \epsilon = \frac{1}{2q}, \text{ while } |x - y| < \delta.$$

It follows that g is discontinuous at x.

For the integrability results, we recall the definition of Riemann sums and the Riemann integral.

Riemann sum. Let $f(x)$ be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$ with $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Denote the width of the ith subinterval $[x_{i-1}, x_i]$ by $\Delta x_i = x_i - x_{i-1}$, and let $\Delta_P = \max_{i=1 \text{ to } n} \Delta x_i$ denote the mesh size of the partition P. If c_i is any point in $[x_{i-1}, x_i]$, then the sum $S = \sum_{i=1}^{n} f(c_i)\Delta x_i$, is called a Riemann sum of $f(x)$ on $[a, b]$ for the partition P.

Riemann integral. The bounded function $f(x)$ on $[a, b]$ is said to be Riemann integrable with value $R = \int_a^b f(x)\,dx$, if for each $\epsilon > 0$, there exists a $\delta > 0$, such that $|S - R| < \epsilon$, for every Riemann sum S of f with $\Delta_P < \delta$.

Now we show that $f(x)$ is not Riemann integrable. We establish this with a proof by contradiction. Assume that $\int_0^1 f(x)\,dx = R$. Let $\epsilon = \frac{1}{2}$. Then there exists a $\delta > 0$, such that $|S - R| < \epsilon$, for every Riemann sum S of f with $\Delta_P < \delta$. We compute two particular Riemann sums S_1 and S_2 for any partition P with $\Delta_P < \delta$. For S_1, choose a rational value $c_i \in [x_{i-1}, x_i]$, for each subinterval in the partition. Then

$$S_1 = \sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} 1\Delta x_i = 1.$$

For S_2, select an irrational value $c_i \in [x_{i-1}, x_i]$, for each subinterval in the partition. Then

$$S_2 = \sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} 0\Delta x_i = 0.$$

However,

$$1 = |S_1 - S_2| \leq |S_1 - R| + |R - S_2| < \frac{1}{2} + \frac{1}{2}$$

yields a contradiction. Thus $f(x)$ is not Riemann integrable.

As for $g(x)$, despite its erratic behavior we can demonstrate that it is Riemann integrable with $\int_0^1 g(x)\,dx = 0$. Consider any value $\epsilon > 0$. Take q large enough so that $\frac{1}{q} < \frac{\epsilon}{2}$. Note that the number of points $N(q)$ in $[0, 1]$ where $g(x) \geq \frac{1}{q}$ is finite. In particular, we have the bound

$$N(q) \leq q + (q-1) + (q-2) + \cdots + 2 + 1 + 1.$$

Consider now $\delta = \frac{\epsilon}{2N(q)}$. Compute the Riemann sum for $g(x)$ over any partition P with $\Delta_P < \delta$ by first classifying the subintervals in P as one of two types. We say a subinterval $[x_{i-1}, x_i] \in P$ is of type 1 if there exists a point $c_i \in [x_{i-1}, x_i]$ for which $g(c_i) \geq \frac{1}{q}$. Otherwise we say a subinterval is of type 2, when $g(c_i) < \frac{1}{q}$, for all $c_i \in [x_{i-1}, x_i]$. Then for any partition P with $\Delta_P < \delta$, we have

$$\begin{aligned} S &= \sum_{\substack{\text{subintervals} \\ \text{of type 1}}} g(c_i)\Delta x_i + \sum_{\substack{\text{subintervals} \\ \text{of type 2}}} g(c_i)\Delta x_i \\ &< (1)N(q)\frac{\epsilon}{2\big(N(q)\big)} + \frac{1}{q}(1-0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $g(x)$ is Riemann integrable.

Part IV

Solutions to Sophisms

7

Functions and Limits

1 Evaluation of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$ proves that $1 = 0$.

The mistake is in part (a). The theorem for the limit of a sum states that

$$\lim_{n\to\infty} \big(f_1(n) + f_2(n)\big) = \lim_{n\to\infty} f_1(n) + \lim_{n\to\infty} f_2(n),$$

provided each of the limits $\lim_{n\to\infty} f_i(n)$ exist for $i = 1, 2$. For any natural number k, we can extend this result to

$$\lim_{n\to\infty} \big(f_1(n) + f_2(n) + \cdots + f_k(n)\big) = \lim_{n\to\infty} f_1(n) + \lim_{n\to\infty} f_2(n) + \cdots + \lim_{n\to\infty} f_k(n),$$

provided each of the limits $\lim_{n\to\infty} f_i(n)$ exist for $i = 1, 2, \ldots, k$. In the example in part (a), k is not a fixed natural number, but is in fact n itself. Therefore the number of terms in our sum is changing with n and we cannot apply the theorem.

2 Evaluation of $\lim_{x\to 0} \left(x \sin \frac{1}{x}\right)$ proves that $1 = 0$.

The mistake is in part (b). If we let $\theta = \frac{1}{x}$, we see that our limit is, in fact, of the form $\lim_{\theta\to\infty} \frac{\sin\theta}{\theta}$ and not of the form $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

3 Evaluation of $\lim_{x\to 0^+}(x^x)$ shows that $1 = 0$.

This time the approach employed in each solution is mathematically flawed! In such expressions it is incorrect to take limits separately in one portion of the expression at a time. However, while both methods presented in (a) and (b) are wrong, it can still be shown using l'Hôpital's rule that 1 is the correct value for the limit. In that case we have,

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{x\ln x}\\ &= e^{\lim_{x\to 0^+} x\ln x}\\ &= e^{\lim_{x\to 0^+} \frac{\ln x}{1/x}}\\ &= e^{\lim_{x\to 0^+} \frac{1/x}{-1/x^2}}\\ &= e^{\lim_{x\to 0^+}(-x)}\\ &= 1.\end{aligned}$$

4 Evaluation of $\lim_{n\to\infty}\sqrt[n]{n}$ demonstrates that $1 = \infty$.

As with the previous sophism it is incorrect to take limits separately one after another in such expressions. Let us find the limit $\lim_{n\to\infty}\sqrt[n]{n}$ using the squeeze theorem. Observe first that $0 = \sqrt[n]{1} - 1 \le \sqrt[n]{n} - 1$.

In the binomial expansion of $n = \left[1 + \left(\sqrt[n]{n} - 1\right)\right]^n$ each term is positive, so it follows that n exceeds any individual term in the expansion. In particular, $n > \binom{n}{2}\left(\sqrt[n]{n} - 1\right)^2$.

This yields the inequality

$$n > \frac{n(n-1)}{2}\left(\sqrt[n]{n} - 1\right)^2.$$

From here we deduce the desired lower and upper bounds:

$$0 < \left(\sqrt[n]{n} - 1\right)^2 < \frac{2}{n-1}.$$

Applying the squeeze theorem gives $\lim_{n\to\infty}\left(\sqrt[n]{n} - 1\right)^2 = 0$ and we therefore conclude $\lim_{n\to\infty}\sqrt[n]{n} = 1$. It is also possible to prove that $\lim_{n\to\infty}\sqrt[n]{n} = 1$, by means of l'Hôpital's rule.

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{(1/n)} &= \lim_{n\to\infty} e^{(\ln n/n)}\\ &= e^{\lim_{n\to\infty}\ln n/n} = e^{\lim_{n\to\infty} 1/n} = 1.\end{aligned}$$

It is worthwhile to note that $\lim_{n\to\infty} \sqrt[n]{n}$ naturally arises in many problems associated with the root test for series.

5 Trigonometric limits prove that $\sin kx = k \sin x$.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, it does not follow that $f(x)$ and $g(x)$ are equal.

6 Evaluation of a limit of a sum proves that $1 = 0$.

Here the mistake is in part (a). We have introduced the same type of error as in Sophism 1. For any k sequences we have

$$\lim_{n\to\infty} \big(s_1(n) + s_2(n) + \cdots + s_k(n)\big) = \lim_{n\to\infty} s_1(n) + \lim_{n\to\infty} s_2(n) + \cdots + \lim_{n\to\infty} s_k(n),$$

provided each of the limits exists. However, in Sophism 6 the value for k is not fixed and varies with n.

7 Analysis of the function $\frac{x+y}{x-y}$ proves that $1 = -1$.

The order of taking limits is important. Although the function is the same, changing the order can give different results. In general,

$$\lim_{x\to a} \lim_{y\to b} f(x, y) \neq \lim_{y\to b} \lim_{x\to a} f(x, y).$$

For example,

$$\lim_{x\to 0} \lim_{y\to 0} y \sin \frac{1}{x} = 0$$

whereas $\lim_{y\to 0} \lim_{x\to 0} y \sin \frac{1}{x}$ is not defined because $\lim_{x\to 0} y \sin \frac{1}{x}$ does not exist.

8 Analysis of the function $\frac{ax+y}{x+ay}$ proves that $a = \frac{1}{a}$, for any value $a \neq 0$.

See the comments in the solution to Sophism 7.

9 One-to-one correspondences imply that $1 = 2$.

Because there are infinitely many points on any line segment and any circumference it doesn't make sense to talk about the number of points. By definition, a set is infinite if there exists a one-to-one correspondence between its elements and the elements of one of its proper subsets. The rules for sets with infinitely many elements are different than the rules for sets with a finite number of elements. In particular, the one-to-one correspondence between the points on these two line segments does not imply that their lengths are the same.

You can readily envision modifications of the construction given in Sophism 9 that would generate a one-to-one correspondence between any two line segments. Perhaps you may be surprised, however, to learn that a one-to-one correspondence between the interval $(0, 1)$ and the real number line $\mathbb{R}$ exists! This correspondence relies on the notion of stereographic projection. We outline the main ideas briefly. First as we already noted, a one-to-one correspondence between the points in the intervals $(0, 1)$ and $(0, 2\pi)$ exists. Define the circle C by $C = \{(x, y)|x^2 + (y - 1)^2 = 1\}$ and let O denote the origin $(0, 0)$ and P denote the point $(0, 2)$. Now observe that a natural one-to-one correspondence between $\theta \in (0, 2\pi)$ and $C(\theta) \in C \sim O$ can be defined as shown in the Figure 7.1.

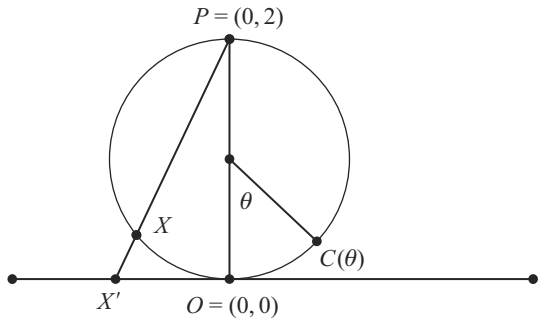


Figure 7.1

Let $X \neq O$ be a point on C. Then the stereographic projection of X is defined as the point X' where the line through P and X intersects the real axis. This projection completes the construction of the one-to-one correspondence between $(0, 1)$ and the real number line $\mathbb{R}$.

You may be even more amazed to discover that one-to-one correspondences exists between points on a unit line segment and points in a unit

square. The correspondences utilized this time rely on the concept of space-filling curves. One can even establish a one-to-one correspondence between points on a 1cm line segment and points of a 3-dimensional figure, for example a sphere of the size of the Earth. As a student remarked, "infinity is where things happen that don't."

For an entertaining introduction to the study of infinity, see Chapter 3 of *The Heart of Mathematics* [5]. More in-depth coverage of the notions of countability and infinity is offered in the text *The Mathematics of Infinity: A Guide to Great Ideas* [10].

10 Aristotle's wheel implies that $R = r$.

This sophism was first described by the Greek philosopher Aristotle (384–322 BC) in his book *Problems of Mechanics*. However, his original explanation was not clear. Later Galileo Galilei (1564–1642) provided his own explanation of the sophism. The mathematical essence of the sophism was only finally fully explained with the development of the concept of one-to-one correspondence between sets of equal cardinality discovered by Georg Cantor (1845–1918). In that regard, the nature of this sophism is similar to Sophism 9(b). To read more about the study of cardinality and infinity we mention again [5, Chapter 3] and [10].

From a practical point of view it is impossible for both wheels to roll. Only the big wheel can roll, and when it does the small wheel both rotates and slides on the surface of the rail. When the big wheel makes one rotation and covers distance AB, point C moves to point D. See Figure 7.2. Distance AB equals the circumference of the big wheel. Obviously $AB = CD$. But CD is bigger than the circumference of the small wheel, because apart from making one rotation the small wheel also slides on the surface of the rail. Point A on the big wheel traces a curve called a cycloid. Point C on the small wheel traces a curve that looks like a flattened cycloid. If the radius of the small wheel is very small, almost zero, then the trajectory of point C will be very close to the straight line OP. In this case the small wheel would be mostly sliding, because the distance CD would be much bigger than the length of the circumference of the small wheel. It is very unlikely that anything would ever be constructed in this way, because the sliding of the small wheel will always create friction.

The small wheel can roll only if the big wheel doesn't contact the surface. Point C on the small wheel traces a cycloid and point A on the big wheel traces a curve that resembles an unfinished circumference. See Figure 7.3. If the radius of the small wheel is very small, almost zero, then

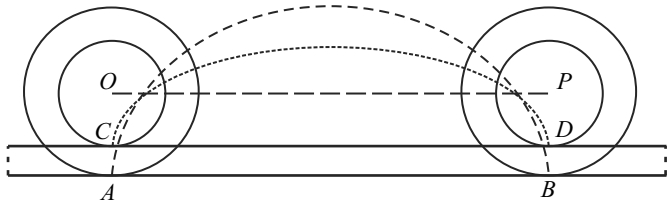


Figure 7.2

the trajectory of point A will be very close to the circumference of the big wheel. For some time during the big wheel's revolution its lowest point will actually move in the opposite direction to that of the overall movement, and there will be a small loop on the bottom of its trajectory.

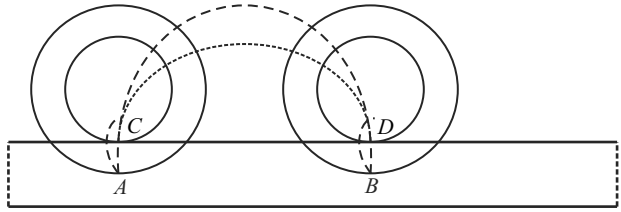


Figure 7.3

This, in fact, is how physical railway wheels move. The small wheel rolls and the big wheel does not touch the surface. See Figure 7.4, which shows a cross-section through the wheel and the rail.

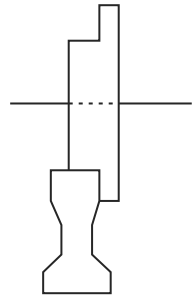


Figure 7.4

11 Logarithmic inequalities show $2 > 3$.

The mistake is in the last step. The value $\ln \frac{1}{2}$ is negative, therefore when dividing by it we need to change the sign of the inequality.

12 Analysis of the logarithm function implies $2 > 3$.

The mistake is in step 2. The function $y = \log_{1/2} x$ is decreasing. Therefore from $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$ it follows that

$$\log_{1/2}\left(\frac{1}{2}\right)^2 < \log_{1/2}\left(\frac{1}{2}\right)^3.$$

13 Analysis of the logarithm function proves $\frac{1}{4} > \frac{1}{2}$.

The mistake is in step 3. The value $\ln \frac{1}{2}$ is negative, therefore when doubling the left-hand side we should instead obtain $2 \ln \frac{1}{2} < \ln \frac{1}{2}$.

14 Limit of perimeter curves shows that $2 = 1$.

As n tends to infinity the length of each line segment approaches zero, but from the diagram we can see that the zig-zag curve tends to the triangle base of length 1. While this is true, it doesn't necessarily mean that the length of the zig-zag curve tends to the length of the base. When n is increasing, the length of each line segment is getting smaller, but the number of segments is getting bigger. At the second stage each line segment length in the zig-zag curve is $\frac{1}{2}$, the number of the line segments is 4, and thus the length of the zig-zag curve is $4 \times \frac{1}{2} = 2$. At the third stage the length of each line segment is $\frac{1}{4}$, the number of the line segments is 8, and therefore the length of the zig-zag curve is $8 \times \frac{1}{4} = 2$. At the nth stage each line segment length is $1/2^{n-1}$ and the number of the segments is 2^n, therefore the length of the zig-zag curve is

$$2^n \times \frac{1}{2^{n-1}} = 2.$$

So when n tends to infinity the limit of the zig-zag curve's length is

$$\lim_{n\to\infty}\left(2^n \times \frac{1}{2^{n-1}}\right) = \lim_{n\to\infty} 2 = 2.$$

When $n \to \infty$ we say that the sequence of curves L_n tends to the segment line L of a finite length if the distance $d\ (L_n, L)$ tends to zero. In our construction if L_n denotes the zig-zag curve at stage n and L denotes

the horizontal base then we have,

$$d(L_n, L) = \frac{\sqrt{3}}{2^n}.$$

Thus by definition $L_n \to L$.

On the other hand, even if $L_n \to L$, this doesn't mean that $\lim_{n\to\infty} l(L_n) = l(L)$. One can only prove that the inequality $\lim_{n\to\infty} l(L_n) \geq l(L)$ holds (provided that the limit $\lim_{n\to\infty} l(L_n)$ exists). In fact, the value $\lim_{n\to\infty} \frac{l(L_n)}{l(L)}$ can be made as large as we desire by modifying the original triangle to be an isosceles triangle with sides of equal length m and base 1. In general, if curve A gets closer to curve B it doesn't mean that the *length* of curve A tends to the *length* of curve B.

15 Limit of perimeter curves shows $\pi = 2$.

The incorrect work lies in part (b). Part (a) has been correctly presented, but as we learned in Sophism 14 even if one curve approaches another, i.e., $L_n \to L$, this doesn't imply that $\lim_{n\to\infty} l(L_n) = l(L)$.

16 Serret's surface area definition proves that $\pi = \infty$.

This is a well-known counterexample to Serret's definition of the area of a surface, found independently by Schwarz in 1880 and by Peano in 1882. This computation showed that Serret's definition of the area of a surface (which was commonly accepted at that time) needed some modifications. This example is known as Schwarz's paradox or the cylinder area paradox. When both m and n tend to infinity the polyhedral surface indeed tends to the lateral surface of the cylinder, but the limit of the area of the polyhedral surface depends on how m and n tend to infinity and can actually be any number larger than or equal to 2π. Serret's definition (quoted below) neglected to consider how m and n tend to infinity:

> "The area of a surface S bounded by a curve C is the limit of the elementary areas of the inscribed polyhedral surfaces P bounded by a curve G as $P \to S$ and $G \to C$, where this limit exists and is independent of the particular sequence of inscribed polyhedral surfaces which is considered."

Similar to the situation that arises in the solution to Sophism 14, if surface A tends to surface B it doesn't imply that the area of surface A tends to the area of surface B.

If you are interested in further information on this paradox, consider the Pólya Award winning article by Frieda Zames [31] on this subject.

17 Achilles and the tortoise

Underlying "The Achilles" sophism of Zeno are deep philosophical questions, much akin to those present in his three other famous sophisms: "The Dichotomy," "The Arrow," and "The Stadium."

> "The Greek philosopher Zeno presented for the first time the problems derived from assuming (or rejecting) the infinite divisibility of space and time. He showed that knowledge of the physical world is dependent on what axioms concerning reality are admitted: space and time are either atomic or dividable ad infinitum." [20]

The interdependence of the concepts of infinity, space, time, and continuity continue to engage mathematicians and philosophers. For an overview of the four "sophisms" of Zeno, consider *History of Mathematics: An Introduction* by Katz, [14, pp. 45–47] or *The History of Mathematics: A Brief Course* by Cooke, [8, pp. 283–284]. Meyerstein offers an engaging account of "The Achilles," in "Is Movement an Illusion? Zeno's Paradox From a Modern View Point," [20].

Here we consider only one aspect of this sophism—a limit. We will not undertake the broader philosophical discussion.

Let the sequence of values a_n represent a set of positions of Achilles and the sequence b_n represent the positions of the tortoise at the corresponding times. Then

$$\begin{aligned} a_1 &= 0 & b_1 &= 1 \\ a_2 &= 1 & b_2 &= 1 + \frac{1}{100} \\ a_3 &= 1 + \frac{1}{100} & b_3 &= 1 + \frac{1}{100} + \frac{1}{100^2} \\ &\vdots & &\vdots \\ a_n &= 1 + \frac{1}{100} + \cdots + \frac{1}{100^{n-2}} & b_n &= 1 + \frac{1}{100} + \cdots + \frac{1}{100^{n-1}} \end{aligned}$$

We have geometric series with the first term 1 and ratio $\frac{1}{100}$. When $n \to \infty$ they have the same limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{1}{1 - \frac{1}{100}} = \frac{100}{99}.$$

This is the position when Achilles will reach the tortoise.

18 Reasonable estimations lead to $1{,}000{,}000 \approx 2{,}000{,}000$.

The rule for multiplying left and right sides of equalities is only valid for exact equalities, and is invalid for approximate equalities. These approximate equalities are in actuality inequalities. For example, $x \approx b$ with accuracy of 0.1 means that $b - 0.1 < x < b + 0.1$ In multiplying left-hand sides and right-hand sides of approximate equalities we actually multiply inequalities of the same nature. For example, from $a < b$ and $c < d$, where each of a, b, c, and $d > 0$, it follows that $ac < bd$. Thus here we are only able to conclude that 1,000,000 < 2,000,000.

19 Properties of square roots prove $1 = -1$.

The property $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ is valid only for nonnegative numbers a and b.

20 Analysis of square roots shows that $2 = -2$.

All three statements are true, but the conclusion is certainly wrong. By definition the number 4 has two square roots, 2 and -2 (the result is 4 when either are squared). The first student mentioned just one of the square roots -2 ("a square root of 4 is -2," which is equivalent to saying "one square root of 4 is -2"). The second student gave the other (nonnegative) square root of 4 because the symbol $\sqrt{\ }$ is used by convention to represent only the nonnegative square root. So the square roots of 4 are 2 and -2 and the equality $\sqrt{4} = 2$ actually reads "the nonnegative square root of 4 is 2," not "the square root of 4 is 2."

21 Properties of exponents show that $3 = -3$.

The exponential rule $(a^m)^n = a^{mn}$ is valid only for nonnegative numbers a.

22 A slant asymptote proves that $2 = 1$.

By definition, a straight line $y = mx + b$ is a slant asymptote to the graph of a function $y = f(x)$ if $\lim_{x\to\infty}\big[f(x) - (mx + b)\big] = 0$. From this definition, it is easy to determine the values of m and b if a slant asymptote exists:

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad b = \lim_{x\to\infty}\big[f(x) - mx\big].$$

For this function, we have

$$m = \lim_{x\to\infty} \frac{x^2 + x + 4}{x(x-1)} = 1$$

and

$$b = \lim_{x\to\infty}\left(\frac{x^2 + x + 4}{x(x-1)} - x\right) = \lim_{x\to\infty}\left(2 + \frac{6}{x-1}\right) = 2.$$

Rewriting $f(x)$ using long division as done in part (a) provides an equivalent means to produce the same values for m and b, when a slant asymptote exists.

The mistake is in part (b). The technique used in (b) only assists in determining that $\lim_{x\to\infty} f(x) = \infty$. Rewriting

$$f(x) = \frac{x^2 + x + 4}{x - 1} = \frac{x + 1 + \frac{4}{x}}{1 - \frac{1}{x}}$$

does not help identify the linear portion of the function.

23 Euler's interpretation of series shows $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$.

In Chapter 20 "Infinite Series" of *Mathematical Thought from Ancient to Modern Times*, Kline [16] provides an illuminating account of the historical development of the theory of series. Numerous delightful examples illustrate the confusion in the early days of this area of mathematics.

The mistake arises in that a series representation is valid only for values of x inside the interval of convergence. Here the interval of convergence of $\sum_0^\infty x^n$ is $(-1, 1)$. The value $x = -1$ lies outside this interval.

24 Euler's manipulation of series proves $-1 > \infty > 1$.

Again as in Sophism 23, the mistake occurs because the series expansions are valid only within their intervals of convergence. The interval of conver-

gence for each of the series $\sum_0^\infty (-1)^n(n+1)x^n$ and $\sum_0^\infty x^n$ is $(-1, 1)$. Thus the evaluations of these series at $x = -1$ and $x = 2$ respectively are both invalid.

This entertaining fallacy of historical note is also discussed in Chapter 20 of [16].

25 A continuous function with a jump discontinuity.

The mistake lies in part (a) within the statement

> "Since each function $f_k(x)$ in the sum is continuous, the sum $S(x)$ is continuous at the values of x where the series converges."

The notion of convergence of functions presented serious conceptual challenges in the development of calculus. The above incorrect statement is actually attributed to Cauchy. Bottazzini [2, pp. 108–112] offers a compelling historical account of this fact and the evolution of the understanding of convergence.

In order to rectify the problem, two types of convergence of functions need to be distinguished. These two notions of pointwise convergence and uniform convergence of functions are typically defined in an advanced calculus or elementary analysis class. We provide these definitions here.

Let $\{S_n(x)\}$ be a sequence of real-valued functions on a set $A \subseteq \mathbb{R}$. We say that $\{S_n(x)\}$ *converges pointwise* to $S(x)$ on A, if $\lim_{n\to\infty} S_n(x) = S(x)$ for all $x \in A$.

Take note of the stronger hypothesis in the definition of uniform convergence.

Let $\{S_n(x)\}$ be a sequence of real-valued functions on a set $A \subseteq \mathbb{R}$. We say that $\{S_n(x)\}$ *converges uniformly* to $S(x)$ on A, if for each $\epsilon > 0$, there exists an N such that $\left|S_n(x) - S(x)\right| < \epsilon$ for all $x \in A$ and all $n > N$.

A treatment of these concepts together with the well-known theorem that the uniform limit of continuous functions is continuous can be found in *Elementary Analysis: The Theory of Calculus* [23]. The moral of the story is to remember that the pointwise limit of a sequence of continuous functions need not be continuous itself.

26 Evaluation of Taylor series proves $\ln 2 = 0$.

Examples of this nature lead to the remark "Occasionally 17th- and 18th-century mathematicians revelled in the art of series-manipulation if for no better reason (it would seem) than to demonstrate their prowess." [15].

In 1837 Dirichlet examined the pair of conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{7.1}$$

and

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} + \cdots \tag{7.2}$$

that are rearrangements of each other [22, pp. 26-31]. He was able to show that the series in (7.1) converges to $\ln 2$, while the series in (7.2) converges to $\frac{3}{2}\ln 2$. This gives yet a third distinct value for what seemingly appears to be the same sum. For the moment we are now facing the sophism $\frac{3}{2}\ln 2 = \ln 2 = 0$!

What Dirichlet had discovered is that convergent series can be classified as one of two distinct types depending on whether or not they remain convergent after all the terms are made positive. Today we refer to these two types of convergence as conditional or absolute convergence of series.

Later in the mid-19th century Riemann's resolution of this seemingly implausible situation was fundamental in the conceptual development of the theory of infinite series. His famous rearrangement theorem for convergent series clarifies that there is nothing special about these three particular values in our sophism!

Theorem. *Given any conditionally convergent series (i.e., one which is not absolutely convergent) and any real number C, there exists a rearrangement of the series so that the newly ordered series converges to C.*

Thus the error in Sophism 26 lies in believing rearrangements of a conditionally convergent series necessarily converge to the same value.

8

Derivatives and Integrals

1 Trigonometric Integration shows $1 = C$, for any real number C.

In both sides of the equation

$$\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2 \tag{8.1}$$

we have the same infinite set of functions written in different forms. For each antiderivative of the left-hand side with a certain value of C_1 we can find the same antiderivative of the right-hand side with another value of C_2 and vice versa. Although being arbitrary, the constants C_1 and C_2 are not independent of each other. They are related by the formula $2C_2 - 2C_1 = 1$. This relationship can be obtained by using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ and replacing $(1 - \sin^2 x)$ with $\cos^2 x$ in (8.1). The difference of two arbitrary but dependent constants is not an arbitrary constant.

2 Integration by parts demonstrates $1 = 0$.

This is similar to the previous sophism. In both sides of the equation

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

we have the same infinite set of functions written in different forms:

$$\ln|x| + C_1 = 1 + \ln|x| + C_2.$$

For each antiderivative of the left-hand side with a certain value of C_1 we can find the same antiderivative of the right-hand side with another value of C_2 and vice versa. Although arbitrary, the constants C_1 and C_2 are *not independent* of each other. They are clearly related by the formula $C_1 = 1 + C_2$.

3 Division by zero is possible.

Although arbitrary, the constants C_1 and C_2 once again are *not independent* of each other. They are related by a formula that can be obtained by applying the rules for logarithms:

$$\begin{aligned}\frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_1 &= \frac{1}{2}\ln|2x+1| + C_2\\ &= \frac{1}{2}\ln\left|2\left(x+\frac{1}{2}\right)\right| + C_2\\ &= \frac{1}{2}\ln 2 + \frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_2.\end{aligned}$$

Thus it follows that $C_1 = \ln\sqrt{2} + C_2$.

While each of C_1 and C_2 can take on the value zero, it just cannot happen at the same time!

4 Integration proves $\sin^2 x = 1$ for any value of x.

After integrating both sides of the differential equation $y'' = (y^2)'$ an arbitrary constant C was omitted. The resulting equation should be $y' = y^2 + C$, or

$$\begin{aligned}\frac{1}{\cos^2 x} &= \tan^2 x + C\\ \sec^2 x &= \tan^2 x + C.\end{aligned}$$

This equation holds true for $C = 1$ and will not lead to an erroneous conclusion.

5 The u-substitution method shows that $\frac{\pi}{2} < 0 < \pi$.

The theorem for the substitution rule for definite integrals is as follows:

Theorem. *Let $u = g(x)$, where $g'(x)$ is continuous on $[a, b]$, and let f be continuous on the range of g. Then*

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

In Sophism 5 we used the substitution $u = \tan x$. Observe that $u' = g'(x) = \sec^2(x)$ is not continuous on $[0, \pi]$, thus the hypothesis for the substitution rule has not been met.

6 $\ln 2$ is not defined.

The mistake is in (a). The function $F(x) = \ln x$ is the antiderivative of $f(x) = \frac{1}{x}$ only for positive values of x. For negative values of x the antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln(-x)$. Combining these two cases we can see that the formula for an antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$ for all nonzero values of x.

7 π is not defined.

Let us state l'Hôpital's Rule.

l'Hôpital's Rule "$\frac{\infty}{\infty}$" case. Suppose that f and g are differentiable on an open interval I of the form (a, ∞) with $g'(x) \neq 0$ on I. If $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists or is $\pm\infty$. In Sophism 7 the limit

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{\pi + \cos x}{1 + \cos x}$$

does not exist, so l'Hôpital's Rule cannot be applied.

8 Properties of indefinite integrals show $0 = C$, for any real number C.

Recall that the property $\int kf(x)\,dx = k\int f(x)\,dx$ is valid only for nonzero values of k. However, the corresponding property of the definite integral is

valid for any value of k. That is,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$$

for all constants k, when $f(x)$ is an integrable function on $[a, b]$.

9 Volumes of solids of revolution demonstrate that $1 = 2$.

The mistake is in (a). We cannot integrate y^2 over the interval $[-2, 2]$ to find the volume of the solid, because the hyperbola does not even exist on the interval $(-1, 1)$, since $x^2 - 1$ is negative for $-1 < x < 1$.

10 An infinitely fast fall

The model implicitly assumes the top of the ladder maintains contact with the wall for the entire duration of the cat's adventure. However, when the angle between the ladder and the x-axis becomes sufficiently small, the action of pulling on the bottom of the ladder will induce the top of the ladder to start pulling away from the wall. From the moment the ladder loses contact with the wall, the right triangle relationship $y = \sqrt{l^2 - x^2}$ is no longer valid. Scholten and Simoson present an entertaining treatise on this problem in "The Falling Ladder Paradox." [30] For the slightly simpler (and possibly more humane) case of the sliding ladder with no cat perched atop, they determine precisely when the top of the ladder will lose contact with the wall. They also analyze the trajectory of the top of the ladder from the time it loses contact with the wall until it crashes to the floor.

11 A positive number equals a negative number.

This example was considered by Cauchy in 1882.

The mistake is in (b). The function $F(x) = \tan^{-1}(\sec x)$ is an antiderivative of the function

$$f(x) = \frac{\sin x}{1 + \cos^2 x}$$

for all points in the interval $\left[0, \frac{3\pi}{4}\right]$ except $x = \frac{x}{2}$. At $x = \frac{\pi}{2}$ the function $F(x) = \tan^{-1}(\sec x)$ is neither differentiable nor continuous. For this

reason we cannot apply the Fundamental Theorem of Calculus to $F(x) = \tan^{-1}(\sec x)$.

One way to find the integral

$$\int_0^{\frac{3}{4}\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

is to consider it on intervals $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{4}]$ and then add the results. Another way is to use the antiderivative $F(x) = -\tan^{-1}(\cos x)$ that is differentiable on $[0, \frac{3\pi}{4}]$. The correct value for the definite integral is $\frac{3\pi}{4} - \tan^{-1}\sqrt{2}$, which is indeed a positive number.

12 The power rule for differentiation proves that $2 = 1$.

For any n functions we have

$$(f_1 + f_2 + \cdots + f_n)'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x),$$

provided that each derivative $f_i'(x)$ exists for $i = 1$ to n.

Here the problem is that n is not a fixed value and varies with x.

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